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The underwater solar light field: analytical model from a WKB evaluation

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ABSTRACT

An analytical expression for the underwater radiance distribution due to a purely "delta function" sun is discussed. The expression derives from a WKB evaluation of the path integral solution for time-dependent radiative transfer, integrated over long times, and does not involve a small-angle approximation. However, a diffusion-limiting length scale previously found in the small-angle approximation also arises in this evaluation, suggesting that it plays a physically important role (independent of approximation schemes) in governing the structure and evolution of the radiance distribution. In its present form, the analytical expression reproduces the shape of the downwelling radiance distribution for angles as large as 90°, but is inadequate for the upwelling component. However, the poor modelling of the upwelling component is not a limitation of the WKB approximation, but is most likely due to the simplistic treatment of the phase function. An effort is underway to more carefully handle the phase function within this WKB framework.

1 INTRODUCTION

Although there exist several sophisticated codes for calculating the radiance distribution in the ocean due to external light sources[1, 2, 3, 4], there are situations when an analytical model is desireable, even if the model is less accurate that the numerical approaches. Toward this end relatively accurate models have been developed which apply only to the up- and down-welling scalar and vector irradiances[5, 6]. The small-angle approximation is a method of generating models of the radiance distribution directly from the radiative transfer equation[7, 8, 9]. While this approximation can model some aspects of the radiance distribution (e.g. the diffuse attenuation coefficient, broadening of the width with depth, and movement with depth of the solar peak angle towards nadir[10]) it is inadequate for modelling the distribution at angles larger than a few degrees from the peak. However, it does have the benefit that, in principle, a systematic perturbation expansion can be constructed to improve its predictive abilities.

In the remaining sections I describe recent work on an analytical approach for directly solving the radiative transfer equation, without requiring the small-angle approximation. The starting point is based on framing the radiative transfer problem as a time-dependent phenomenon, with an external power source that is constant in time. If we begin from a condition of no radiance, the radiance ramps up over a short time period due to the constant power source, and the long time result is the solution of the time-independent

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radiative transfer problem. The reason for adding the extra time-dependent behavior is that it allows certain simplifications in the mathematical formulation, and in particular allows us to abandon the small-angle approach. The outcome is a formally exact solution in terms of a phase-space path integral over all paths light rays can take from the source to the observation point. The steps to this formalism are presented in section 2.

An approximate analytical expression is derived in section 3 using the WKB approximation for path integrals, and Laplace's approximation for the integral over time. These approximations have impacts on the physical fidelity of the result in several ways, some of which are discussed. An interesting consequence of this approach is that a length scale ℓ, defined as

$$\ell = \frac{1}{\sqrt{\langle \theta^2 \rangle ab}} ,$$

arises in the analytical result. This length scale also appears in the small-angle approximation[9], and could be interpreted in several ways. However, because it appears in both the small-angle and WKB approximations, ℓ may be a fundamental controlling parameter of the radiance distribution.

In order to assess the success of the WKB approach, the analytical expression is compared in section 4 with the radiance data collected by Tyler[11]. The shortcomings of the current results are discussed, and steps for improvement suggested.

The discussion here is restricted to the case of a uniform unstratified ocean, with scattering coefficient b and absorption coefficient a.

2 TIME-DEPENDENT PATH INTEGRAL FORMULATION

The mathematical problem of obtaining the radiance distribution in the ocean due to solar lighting is usually approached as a boundary value problem, in which the radiative transfer equation

$$\{\hat{n} \cdot \nabla + c\} \; L(\vec{x}, \hat{n}) = b \int d\Omega' \; P(\hat{n} \cdot \hat{n}') \; L(\vec{x}, \hat{n}')$$

is solved subject to boundary conditions, including the condition that the downwelling radiance at the surface is some prescribed form due to solar illumination. In the approach used here, the solar illumination is treated as an external source at the ocean surface ($z = 0$), and the radiative transfer equation is now

$$\{\hat{n} \cdot \nabla + c\} \; L(\vec{x}, \hat{n}) - b \int d\Omega' \; P(\hat{n} \cdot \hat{n}') \; L(\vec{x}, \hat{n}') = F_{solar}(\vec{x}, \hat{n}) \; .$$

For a "delta-function" sun used here,

$$F_{solar}(\vec{x}, \hat{n}) = F_{solar} \; \delta(z) \; \delta(n - n_{sun}) \; ,$$

where $\hat{n}_{sun}$ is the direction the sun rays are pointing into the water.

If an operator $g(\vec{x}, \hat{n}; \vec{x}', \hat{n}')$ is defined as

$$g(\vec{x}, \hat{n}; \vec{x}', \hat{n}') = \{[\hat{n} \cdot \nabla + c] \; \delta(\hat{n} - \hat{n}') - b \; P(\hat{n} \cdot \hat{n}')\} \; \delta(\vec{x} - \vec{x}')$$

then the radiative transfer equation can be phrased as a multidimensional integral equation

$$\int d^3x'\, d\Omega'\, g(\vec{x},\hat{n};\vec{x}',\hat{n}')\, L(\vec{x}',\hat{n}') \;=\; F_{solar}(\vec{x},\hat{n})\;.$$

A formal solution of the form

$$L(\vec{x},\hat{n}) \;=\; \int d^3x'\, d\Omega'\, g^{-1}(\vec{x},\hat{n};\vec{x}',\hat{n}')\, F_{solar}(\vec{x}',\hat{n}')$$

follows from the construction of the inverse operator g^{-1}, subject to boundary conditions.

An alternative, mathematically equivalent, solution comes from introducing the time-dependent radiative transfer problem

$$\left\{\frac{\partial}{\partial s} + \hat{n}\cdot\nabla + c\right\} L(s,\vec{x},\hat{n}) - b\int d\Omega'\, P(\hat{n}\cdot\hat{n}')\, L(\vec{x},\hat{n}') = F_{solar}(\vec{x},\hat{n})\;, \tag{1}$$

where $s = vt$ is time in units of length by scaling with the speed of light in water v. The solution of this problem can be written with the help of a kernel operator G as

$$\begin{aligned} L(s,\vec{x},\hat{n}) \;&=\; \int d^3x'\, d\Omega'\, G(s,\vec{x},\hat{n};\vec{x}',\hat{n}')\, L(0,\vec{x}',\hat{n}') \\ &+\; \int_0^s ds' \int d^3x'\, d\Omega'\, G(s-s',\vec{x},\hat{n};\vec{x}',\hat{n}')\, F_{solar}(\vec{x}',\hat{n}')\;, \end{aligned} \tag{2}$$

where $L(0,\vec{x},\hat{n})$ is the initial radiance distribution, and G satisfies equation 1 with $F_{solar} = 0$ and has the initial condition

$$G(0,\vec{x},\hat{n};\vec{x}',\hat{n}') = \delta(\vec{x}-\vec{x}')\,\delta(\hat{n}-\hat{n}')\;. \tag{3}$$

Using the operator g, G can be written

$$G = (\exp\{-sg\})\,.$$

From this form, a path integral expression for G was previously found as[12]

$$\begin{aligned} G(s,\vec{x},\hat{n};\vec{x}',\hat{n}') \;&=\; e^{-cs}\int [d^3p]\,[d\Omega]\,\delta\left(\hat{\beta}(0)-\hat{n}'\right)\,\delta\left(\hat{\beta}(s)-\hat{n}\right)\,\delta\left(\vec{x}-\vec{x}'-\int_0^s ds'\,\hat{\beta}(s')\right) \\ &\times\; \exp\left\{b\int_0^s ds'\, Z(\vec{p}(s'))\right\}\,\exp\left\{i\int_0^s ds'\,\vec{p}(s')\cdot\frac{d\hat{\beta}(s')}{ds'}\right\}\;, \end{aligned} \tag{4}$$

where $Z(\vec{p})$ is the "pseudo"-Fourier transform of the phase function, and $\hat{\beta}(s)$ is the path a lightray can take from the point $\vec{x}'$ to the observation point $\vec{x}$.

To obtain the time-independent distribution, we take the $s \to \infty$ limit of equation 2, to arrive at

$$L(\vec{x},\hat{n}) = \int_0^\infty ds \int d^3x'\, d\Omega'\, G(s,\vec{x},\hat{n};\vec{x}',\hat{n}')\, F_{solar}(\vec{x}',\hat{n}')\;. \tag{5}$$

Two additional steps allow further reduction of equation 5 before approximations must be employed. One of them uses the fact that G is of function of $\vec{x}$ and $\vec{x}'$ only in the form $\vec{x}-\vec{x}'$ (see equation 4). Therefore the convolution over $\vec{x}'$ can be done in Fourier space, using the Fourier-transformed quantity

$$\tilde{G}(s,\vec{q},\hat{n},\hat{n}') = \int d^3x\; G(s,\vec{y}+\vec{x},\hat{n};\vec{y},\hat{n}')\,\exp\{-i\vec{q}\cdot\vec{x}\}\;. \tag{6}$$

The second step uses the representation of the delta-function sun. The total result is

$$L(z,\hat{n}) = F_{solar}\int_0^\infty ds \int_{-\infty}^\infty \frac{dq_z}{2\pi}\,\tilde{G}(s,q_z,\hat{n},\hat{n}_{sun})\exp\{iq_z z\}\;, \tag{7}$$

with q_z being the z-compnent of $\vec{q}$.

3 ANALYTICAL APPROXIMATION

Equations 4 and 7 are the necessary ingredients to begin the WKB evaluation outlined in the next section. The path integral in equation 4 will be evaluated by the WKB procedure[13, 14], and the integral over time in equation 7 will be evaluated by Laplace's method[15], i.e. by expanding the integrand as a gaussian around a maximum point. The final expression for the radiance is equation 13 below, with terms defined in the discussion surrounding it.

3.1 WKB Evaluation

There are two functional integrals in equation 4 which must be approximately evaluated. The integral over the Fourier transform variable $\vec{p}(s)$ is evaluated first, followed by the integral over lightray paths $\hat{\beta}(s)$.

The integral of $\vec{p}(s)$:

$$\int [d^3p] \; \exp\left\{ \int_0^s ds' \; \left(bZ(\vec{p}(s')) + i\vec{p}(s') \cdot \frac{d\hat{\beta}(s')}{ds'} \right) \right\}$$

cannot be evaluated for general forms of Z. For a sharply forward-peaked phase function, the form frequently used in the small-angle approximation is

$$Z(\vec{p}) \approx 1 - \frac{\langle \theta^2 \rangle}{2} p^2 \;, \tag{8}$$

where $\langle \theta^2 \rangle$ is the mean square width of the forward peak. This form is also used here, recognizing that it is an inadequate treatment of backscatter. In fact, this one approximation probably accounts for the underestimate of upwelling radiance shown in section 4 below. Work is underway to handle more reasonable forms for Z. Nevertheless, we can expect the WKB evaluation to succeed at modelling the radiance distribution better than the small-angle approximation, because the additional small-angle assumption that $\hat{n}$ and $\hat{n}_{sun}$ are both close to nadir is *not* used here.

Having adopted equation 8, the functional integral over $\vec{p}$ can be evaluated because it is gaussian, to give ($\dot{\hat{\beta}} \equiv d\hat{\beta}/ds'$)

$$\exp\left\{ -\frac{1}{2b\langle \theta^2 \rangle} \int_0^s ds' \; \left| \dot{\hat{\beta}}(s') \right|^2 \right\} .$$

Here I have ignored normalizing constants, which can be picked up again in the final result when the initial condition in equation 3 is applied.

After Fourier transforming as in equation 6, the remaining integral is

$$\begin{aligned} \tilde{G}(s, \vec{q}, \hat{n}, \hat{n}') &= e^{-as} \int [d\Omega] \; \delta\left(\hat{\beta}(0) - \hat{n}_{sun}\right) \; \delta\left(\hat{\beta}(s) - \hat{n}\right) \\ &\times \; \exp\left\{ -\frac{1}{2b\langle \theta^2 \rangle} \int_0^s ds' \; \left| \dot{\hat{\beta}}(s') \right|^2 \right\} \; \exp\left\{ -i \int_0^s ds' \; \vec{q} \cdot \hat{\beta}(s') \right\} . \end{aligned} \tag{9}$$

As a sidenote, this path integral has an exact solution for $\vec{q} = 0$ (see reference 16):

$$\tilde{G}(s, 0, \hat{n}, \hat{n}') = e^{-as} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \mathrm{Y}_{nm}(\hat{n})\ \mathrm{Y}^{*}_{nm}(\hat{n}')\ \exp\left\{ -\frac{b\langle\theta^2\rangle s}{2} n(n+1) \right\} ,$$

where the Y_{nm} are the spherical harmonics. This particular result might prove useful as a limiting test case in future efforts to develop improved models, but currently is not used.

To carry out the WKB approximation, the integration function $\hat{\beta}(s')$ is decomposed into two components: $\hat{\beta}(s') = \hat{\beta}_0(s') + \vec{\gamma}(s')$, where $\hat{\beta}_0(s')$ is the mean path for the dominant contribution to the path integral, and $\vec{\gamma}$ represents small deviations around $\hat{\beta}_0$. The mean path is obtained by minimizing the effective attenuation

$$\frac{1}{2b\langle\theta^2\rangle} \int_0^s ds' \left| \dot{\hat{\beta}}(s') \right|^2 ,$$

subject to the boundary conditions

$$\begin{aligned} \hat{\beta}_0(0) &= \hat{n}_{sun} , \\ \hat{\beta}_0(s) &= \hat{n} , \end{aligned}$$

and the condition that $\hat{\beta}_0$ is a unit vector. Expressing $\hat{\beta}_0$ in terms of polar coordinates $(\vartheta(s'), \varphi(s'))$, the Euler-Lagrange equations for minimizing the attenuation are

$$\begin{aligned} \ddot{\vartheta}(s') - \dot{\varphi}^2(s') \sin\vartheta(s') \cos\vartheta(s') &= 0 \\ \frac{d}{ds'} \left[\dot{\varphi}(s') \sin^2\vartheta(s') \right] &= 0 . \end{aligned}$$

These equations have the general solution

$$\begin{aligned} \cos\vartheta(s') &= \cos\alpha\ \cos\left(D(s' - s_0)\right) \\ \varphi(s') &= \varphi_0 + \arctan\left\{ \frac{\tan\left(D(s' - s_0)\right)}{\sin\alpha} \right\} , \end{aligned}$$

where α, s_0, φ_0, and D are integration constants. In addition, the Euler-Lagrange equations also give

$$\left| \dot{\hat{\beta}}(s') \right|^2 = \dot{\vartheta}^2(s') + \dot{\varphi}^2(s') \sin^2\vartheta(s') = D^2 = \text{constant} ,$$

so that

$$\exp\left\{ -\frac{1}{2b\langle\theta^2\rangle} \int_0^s ds'\ \left| \dot{\hat{\beta}}(s') \right|^2 \right\} = \exp\left\{ -\frac{D^2\ s}{2b\langle\theta^2\rangle} \right\} .$$

Applying the boundary conditions gives

$$D = \frac{\Theta}{s} ,$$

where $\Theta = \arccos(\hat{n} \cdot \hat{n}_{sun})$ is the angle between the observing and solar directions. In addition,

$$\begin{aligned} \sin\alpha &= \sin\theta_{sun}\ \sin(\phi - \phi_{sun}) \\ \sin(Ds_0) &= \frac{\cos\theta - \cos\Theta \cos\theta_{sun}}{\cos\alpha \sin\Theta} \\ \varphi_0 &= \phi_{sun} + \arctan\left\{ \frac{\tan(Ds_0)}{\sin\alpha} \right\} . \end{aligned}$$

In these expressions, (θ, ϕ) and $(\theta_{sun}, \phi_{sun})$ are the polar coordinates for $\hat{n}$ and $\hat{n}_{sun}$ respectively.

Using the decomposition of $\hat{\beta}$ into $\hat{\beta}_0$ and $\vec{\gamma}$, and setting the horizontal components of $\vec{q}$ to zero, we arrive at

$$\begin{aligned} \tilde{G}(s, q_z, \hat{n}, \hat{n}_{sun}) &= \exp\left\{-as - \frac{\Theta^2}{2b\langle\theta^2\rangle s} - iq_z s\xi\right\} \\ &\times \int[d\gamma]\, \delta(\vec{\gamma}(0))\, \delta(\vec{\gamma}(s)) \exp\left\{-\int_0^s ds' \left[\frac{\left|\dot{\vec{\gamma}}(s')\right|^2}{2b\langle\theta^2\rangle} + iq_z\hat{z}\cdot\vec{\gamma}(s')\right]\right\}, \end{aligned} \tag{10}$$

where

$$\xi = \frac{1}{s}\int_0^s ds' \cos\vartheta(s') = \frac{1-\cos\Theta}{\Theta\sin\Theta}(\cos\theta + \cos\theta_{sun}) .$$

This expression is still exact, because the decomposition does not constitute an approximation. The WKB approach however, tells us to treat $\vec{\gamma}(s')$ as a "small" deviation from the mean path $\hat{\beta}_0(s')$. To do that carefully, the constraint that $\hat{\beta}$ be a unit vector must be maintained in some way. This constraint can be written as

$$\gamma^2(s') + 2\vec{\gamma}(s')\cdot\hat{\beta}_0(s') = 0 .$$

While a careful evaluation using this constraint is desireable, for now it will be ignored. The integral in 10 then has a gaussian form, to give

$$\int[d\gamma]\, \delta(\vec{\gamma}(0))\, \delta(\vec{\gamma}(s)) \exp\left\{-\int_0^s ds' \left[\frac{\left|\dot{\vec{\gamma}}(s')\right|^2}{2b\langle\theta^2\rangle} + iq_z\hat{z}\cdot\vec{\gamma}(s')\right]\right\} \approx s^{-1}\exp\left\{-6b\langle\theta^2\rangle q_z^2 s^3\right\} \tag{11}$$

up to overall normalization. The integral over q_z in equation 7 can now be evaluated since it is now gaussian:

$$L(z, \hat{n}) \approx F_{solar}\int_0^\infty ds\; s^{-5/2}\exp\left\{-as - \frac{\Theta^2}{2b\langle\theta^2\rangle s} - \frac{(z - s\xi)^2}{24b\langle\theta^2\rangle s^3}\right\} . \tag{12}$$

3.2 Time Integral Evaluation

The remaining integral over time s in equation 12 cannot be evaluated exactly. However, in comparing a direct numerical integration of a few cases with the result of evaluation by Laplace's method below, Laplace's method succeeded in reasonably reproducing the more accurate numerical result. Certainly more important errors arise from the assumption in equation 8 for the phase function.

Before carrying out the approximate evaluation, the particular case $z = 0$ can be evaluated exactly, to give (see integral 3.471.9 of reference 17)

$$L(0, \hat{n}) = F_{solar}\left[\frac{1}{2}\left(\Theta^2 + \xi^2/12\right)\ell^2\right]^{-3/4} K_{3/2}\left(\sqrt{2\left(\Theta^2 + \xi^2/12\right)}\ell a\right) ,$$

where $\ell^2 = 1/(\langle\theta^2\rangle ba)$ is the length scale found in the small-angle approximation, and $K_{3/2}$ is a modified Bessel function. This solution is plotted in figure 1 for $\phi = \phi_{sun}$ and $\theta_{sun} = 24°$, and is dominated by upwelling light. This is because the light source is at $z = 0$, and so the only downwelling light coming into

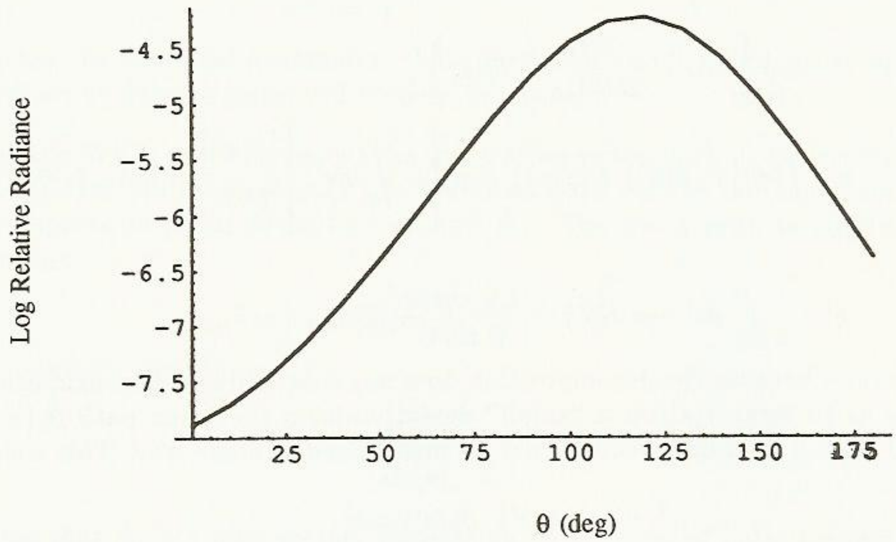

Figure 1: Radiance at the $z = 0$ surface in the plane of the sun. The sun is located at $\theta_{sun} = 24°$.

the $z = 0$ plane from above is from backscatter of the upwelling light. Just below the $z = 0$ plane, there is a downwelling peak in the $\hat{n} = \hat{n}_{sun}$ direction.

For $z > 0$ the integral in equation 12 does not have a known general solution. The approach taken here is to use Laplace's method to approximately evaluate the integral. Writing

$$L(z, \hat{n}) = F_{solar} \int_0^\infty ds\ s^{-5/2} \exp\{-S(s)\} ,$$
$$S(s) = as + \frac{\Theta^2}{2b\langle\theta^2\rangle s} + \frac{(z - s\xi)^2}{24b\langle\theta^2\rangle s^3} ,$$

the integrand has a maximum where S has a minimum. That point s_0 satisfies the quartic equation ($\tau = s_0/\ell$; $x = z/\ell$)

$$\tau^4 - \tau^2 \left(\frac{\Theta^2}{2} + \frac{\xi^2}{24} \right) + \tau \frac{x\xi}{4} - \frac{x^2}{8} = 0 ,$$

which has only one real solution that minimizes S. Expanding about s_0,

$$S(s) \approx S(s_0) + \frac{1}{2}(s - s_0)^2\ S''(s_0) ,$$

with

$$S''(s_0) = \frac{1}{b\langle\theta^2\rangle s_0^3} \left[\Theta^2 + \frac{\xi^2}{12} + \frac{(z - s_0\xi)^2}{2s_0^2} + \frac{\xi(z - s_0\xi)}{2s_0} \right] .$$

The integral remaining is now gaussian, giving

$$L(z,\hat{n}) = F_{solar}\ \{S''(s_0)\}^{-1/2}\ s_0^{-5/2}\ \exp\{-S(s_0)\}\ . \tag{13}$$

Note that because S has the form

$$S(s) = as + \frac{F(s)}{b\langle\theta^2\rangle s}\ ,$$

with F a dimensionless function independent of optical properties, the solution scales as $s_0 = \ell\tau$, where τ is dimensionless. Just as in the exact integral evaluation at $z = 0$, and as in the small-angle approximation, this approximate evaluation identifies the length scale ℓ as an important scale parameter.

4 COMPARISON WITH DATA

As a first evaluation of the outcome of equation 13, the predicted radiance distribution is shown in figure 2, along with data reported by Tyler[11] from Lake Pend Oreille. The figure shows radiance in the plane of the sun at 7 depths. The absorption and scattering coefficients used in the model are $a = 0.12\ \mathrm{m}^{-1}$ and $b = 0.28\ \mathrm{m}^{-1}$, as reported by Tyler. The model radiance is normalized to the data value at $z = 66.1$ m, $\theta = 0°$. The phase function width $\langle\theta^2\rangle$ was set to the value 0.035. This value provides a qualitatively best fit to the data, and was also previously found to be the best in a comparison of the same data with the small-angle approximation[10].

Although the model clearly underestimates the upwelling light, the shape of the distribution for $\theta > 90°$ is more reasonable at depths below about 10 m ($\sim$ 4 beam attenaution lengths). There are three ways in which improvements in the analytical approach should improve the comparison:

1. As mentioned earlier, a better representation of the phase function is needed in order to increase the magnitude of backscatter in the model. Work is underway to allow general phase functions, at the cost perhaps of needing some numerical iteration to solve a transcendental equation.

2. The solar distribution just above the ocean surface includes a relatively smooth background light field in addition to the direct sunlight. The background appears to be important in forming the upwelling light field near the surface.

3. The "sloppy" evaluation of the path integral over $\vec{\gamma}$ in equation 11 should be corrected. This will be particulary important if the solution were used in time-dependent problems, such as pulse propagation.

Despite these shortcomings of equation 13, the result here is an improvement over the small-angle approximation in predicting the radiance distribution. In addition, the WKB approach provides a general and systematic scheme for removing the physical and mathematical approximations used previously in deriving solutions of the radiative transfer equation.

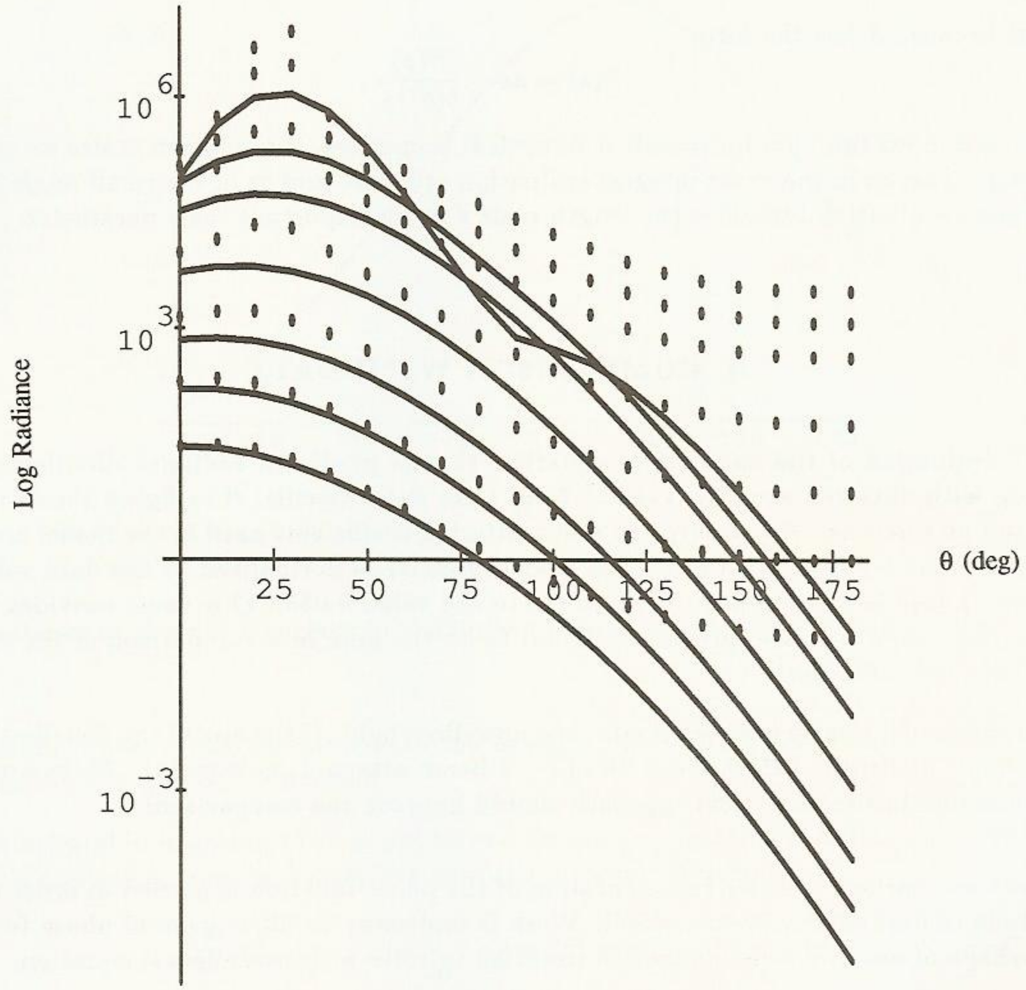

Figure 2: Radiance in the plane of the sun at depths of 4.24, 10.4, 16.6, 29.0, 43.1, 53.7, and 66.1 meters. The dots are data points from Tyler.

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