Radiative transfer as a sum over paths

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The radiative-transfer equation describes the collection of paths taken by an element of radiation as it travels from one location to another. When backscatter can be ignored, the exact solution is constructed as a formal sum (path integral) over all such paths. In the appropriate limit the usual (diffusive) small-angle solution and the multiple-scattering solution can be obtained. Another small-angle solution has also been found which includes some of the nonlinear and large-angle behavior not present in the diffusive solution. After several attenuation lengths, length scales are characterized by a parameter constructed out of the absorption and scattering coefficients, and the rms scattering angle per scattering event. The two solutions are compared in the case of a point beam.

I. INTRODUCTION

The radiative-transfer (RT) equation has been extensively studied, both analytically and numerically. Accurate solutions for a variety of boundary and initial data problems have been obtained using Monte Carlo[1] and finite-difference[2] numerical approaches, but it has been very difficult to find analytic solutions valid for a broad collection of phase functions and initial data/boundary conditions. Wells[3] has developed a spherical-harmonic expansion valid for general phase functions and angular distributions, but limited to a slab geometry. The low terms of the expansion are determined from coupled differential-difference equations, but for many practical situations higher-order terms can be approximated by an asymptotic form and the expansion can be summed. The small-angle approximation[4] provides a solution for (apparently) arbitrary spatial distributions, but the phase function must be sharply forward peaked and the radiance confined to a narrow range around the axis of propagation. The solution is maximally diffusive in the sense that it contains more scattering events per path length than any other solution. Stotts[5] has argued that this solution can be valid only after many (>10) scattering lengths have been traversed because it suppresses small-scale spatial features that may be present in the initial data, and because it does not contain multiple-scattering effects. Multiple-scattering events may be considered those which scatter through angles much larger than the rms scattering angle of the forward-peaked phase function. Stotts attributes multiple-scattering processes to the higher harmonic terms ignored by the small-angle solution. The source of higher harmonics in Stott's argument, however, is the Fourier-transformed phase function. His analysis does not mention the higher-order absorption and spatial terms left out of the small-angle form.

A question arises from this: What are the effects of the ignored angular dependence in the absorption and spatial terms? It might be expected that the absorption term confines the radiance to a smaller angular range than if it were not present. But if the phase function is sufficiently forward peaked, is it possible that the scattering will be confined to such a narrow range that the extra absorption effects can be ignored and the diffusive small-angle solution remains valid? Including higher-order angular terms in the spatial behavior of the RT equation gives a greater lever arm to a ray of radiation passing from some initial location to the observation point. Consequently each observation point can receive radiance from a larger range of distances, apparently smoothing the spatial distribution more than in the small-angle solution. This is deceiving, however, because the high-order angular contributions from the scattering and absorption terms strongly suppress the importance of radiance at large angles to the direction of observation, possibly improving spatial resolution. The coupling between the spatial and angular degrees of freedom may be subtle, and it would be useful to know more precisely how resolution is improved over the small-angle solution by high-order terms.

The question in the last paragraph of spatial and angular coupling will not be discussed further. The question of higher-order absorption effects is discussed in Sec. V, where a small-angle solution is found which includes absorption contributions to the distribution. The overall conclusion reached there is that as long as the rms scattering angle is not zero, the radiance distribution far from the initial data plane loses the diffusive form and includes some multiple-scattering properties. The distribution is not broader than the small-angle solution—as might be expected from the increased importance of multiple-scattering events—but becomes narrower as the radiance proceeds away from the initial data plane. With the increased freedom to undergo large-angle scattering, the distribution tends to follow the initial distribution farther than allowed by the small-angle solution, scattering in the observation direction closer to the point of observation. The transition from the small-angle (SA) solution to this "small-angle with absorption" (SAA) solution is characterized by a length scale l determined by the scattering and absorption processes.

Before focusing on the SAA solution in Sec. V. Section II outlines an exact solution of the radiative-transfer equation. It is valid for forward scattering in an initial data problem. The solution is formally a path integral, in

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which integration represents a sum over all possible angular paths the radiation may take in traversing from the initial data plane to the observation point. The approach is based on some methods developed by Fradkin and others which have been applied to problems in quantum field theory, potential theory,[6] and the Navier-Stokes equations.[7] The well-known multiple-scattering and small-angle solutions are derived from the general solution in Secs. III and IV using appropriate approximations.

Before outlining the solution, the scattering geometry and notation should be defined. We assume that at $z=0$ a radiance distribution $I_0(\mathbf{x},\hat{\mathbf{n}})$ is known, where $\mathbf{x}$ is a two-dimensional (2D) spatial vector in the $z=$const plane, and $\hat{\mathbf{n}}$ the direction vector of the radiance scattered into polar angles (θ,ϕ) $(0\leq\theta<\pi, 0\leq\phi<2\pi)$. For $z>0$, the RT equation gives the radiance distribution $I(\mathbf{r},\hat{\mathbf{n}})$ at the 3D position $\mathbf{r}=(\mathbf{x},z)$ traveling in the direction $\hat{\mathbf{n}}$. For forward scattering only, we restrict $\hat{\mathbf{n}}$ such that $0\leq\theta\leq\pi/2$. The RT equation is

$$(\hat{\mathbf{n}}\cdot\nabla_{\mathbf{r}}+c)I(\mathbf{r},\hat{\mathbf{n}})=b\int d^2n'\,S(\hat{\mathbf{n}},\hat{\mathbf{n}}')I(\mathbf{r},\hat{\mathbf{n}}')\;, \tag{1}$$

where b is the scattering coefficient, $a=c-b$ is the absorption coefficient, and S is the phase function normalized to unity:

$$\int d^2n'\,S(\hat{\mathbf{n}},\hat{\mathbf{n}}')=1\;.$$

II. SOLUTION

The explicit rotational symmetry of the RT equation is broken when a solution is constructed on the basis of initial data, as would be the case, for example, in a slab geometry. We assume that a three-dimensional spatial coordinate system with components $\mathbf{r}=(\mathbf{x},z)$ is used, and the unit vector $\hat{\mathbf{n}}$ is restricted to the hemisphere $0<\theta<\pi/2$ with $z>0$. This allows the decomposition $\hat{\mathbf{n}}=(\mathbf{n},\eta(n))$, where $\eta(n)=(1-\mathbf{n}^2)^{1/2}$. In spherical coordinates

$$\mathbf{n}=\sin\theta(\cos\phi,\sin\phi)\;,$$

$$\eta(n)=\cos\theta\;.$$

The initial data are the radiance $I_0(\mathbf{x},\mathbf{n})$ at $z=0$. With this information the solution of the RT equation is expressed in terms of the point spread function (PSF) $G(\mathbf{x},\mathbf{n},\mathbf{n}',z)$ by

$$I(\mathbf{x},\mathbf{n},z)=\int d^2x'\int d^2n'\,G(\mathbf{x}-\mathbf{x}',\mathbf{n},\mathbf{n}',z)I_0(\mathbf{x}',\mathbf{n}')\;, \tag{2}$$

with G satisfying the initial condition

$$G(\mathbf{x},\mathbf{n},\mathbf{n}',z=0)=\delta(\mathbf{x})\delta(\mathbf{n}-\mathbf{n}')\;. \tag{3}$$

In the chosen coordinate system, the RT equation is

$$\left[\eta(n)\frac{\partial}{\partial z}+\mathbf{n}\cdot\nabla_{\mathbf{x}}+c\right]G(\mathbf{x},\mathbf{n},\mathbf{n}',z)=b\int d^2n''\,S(\mathbf{n},\mathbf{n}'')G(\mathbf{x},\mathbf{n}'',\mathbf{n}',z)\;. \tag{4}$$

The principle difficulty in solving this equation is handling the integral on the right-hand side. This term can be converted to a nonlocal differential term by a "pseudo-Fourier" representation of the phase function:

$$S(\mathbf{n},\mathbf{n}')=\int\frac{d^2p}{(2\pi)^2}\tilde{S}(\mathbf{n},\mathbf{p})\exp[i\mathbf{p}\cdot(\mathbf{n}-\mathbf{n}')]\;. \tag{5}$$

This expansion is not intended as a true Fourier decomposition with an inverse transform because the angular vector $\mathbf{n}$ is restricted to the compact space within the unit disk. Using Eq. (5) and a spatial Fourier transform of the PSF to the modulation transfer function (MTF)

$$G(\mathbf{x},\mathbf{n},\mathbf{n}',z)=\int\frac{d^2k}{(2\pi)^2}\tilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z)\exp(i\mathbf{k}\cdot\mathbf{x})\;,$$

the RT equation becomes[8]

$$\left[\frac{\partial}{\partial z}+i\frac{\mathbf{k}\cdot\mathbf{n}}{\eta(n)}+\frac{c}{\eta(n)}-\frac{b}{\eta(n)}S(\mathbf{n},-i\nabla_n)\right]\tilde{G}=0\;. \tag{6}$$

The initial condition now takes the form

$$\tilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z=0)=\delta(\mathbf{n}-\mathbf{n}')\;. \tag{7}$$

Formally, the solution of Eqs. (6) and (7) can be written in terms of an exponential:

$$\tilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z)=\exp\left[-z\left[L(\mathbf{k},\mathbf{n})-\frac{b}{\eta(n)}\tilde{S}(\mathbf{n},-i\nabla_n)\right]\right]\delta(\mathbf{n}-\mathbf{n}')\;, \tag{8}$$

where

$$L(\mathbf{k},\mathbf{n})=i\frac{\mathbf{k}\cdot\mathbf{n}}{\eta(n)}+\frac{c}{\eta(n)}\;.$$

This can be converted to a path integral form by following several steps. First, introduce an auxiliary function $\mathbf{Q}(z)$, which will eventually be set to zero, and a functional of $\mathbf{Q}$, denoted $\tilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z\mid\mathbf{Q})$, which satisfies the pair of equations

$$\begin{aligned}&\left[\frac{\partial}{\partial z}+L(\mathbf{k},\mathbf{n})+\mathbf{Q}\cdot\nabla_n-\frac{b}{\eta(n)}\tilde{S}\left[\mathbf{n},i\frac{\delta}{\delta\mathbf{Q}(z)}\right]\right]\tilde{G}(\mathbf{Q})=0\;,\\&\frac{\delta}{\delta\mathbf{Q}(z)}\tilde{G}(\mathbf{Q})=-\nabla_n\tilde{G}(\mathbf{Q})\;,\end{aligned} \tag{9}$$

and the initial condition (7). The solution of the RT equation sought is $\tilde{G}(\mathbf{Q}=\mathbf{0})$. The introduction of $\mathbf{Q}(z)$ allows us to use the path integral identity [$A(z)$ and $B(z)$ are arbitrary functions]

$$\begin{aligned}F(A)&=\int[DB]\prod_{z'=0}^{z}\delta(A(z')-B(z'))F(B)\\&=N^{-1}\int[DB][D\phi]F(B)\\&\quad\times\exp\left[i\int_0^z dz'\,\phi(z')(A(z')-B(z'))\right]\end{aligned}$$

(the measure $[DB]$ is $\prod_{z'=0}^{z}dB(z')$) on the ordered exponential solution of Eq. (9) [which is analogous to equation (8)] to obtain

$$\widetilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z\mid\mathbf{Q})$$
$$=N^{-1}\int[D\Pi][D\phi]\exp\left[i\int_0^z dz'\,\phi(z')\cdot\Pi(z')\right]$$
$$\times\exp\left[\int_0^z dz'\,\phi(z')\cdot\frac{\delta}{\delta\mathbf{Q}(z')}\right]\left\{\exp\left[-\int_0^z dz'\left[L(\mathbf{k},\mathbf{n})+\mathbf{Q}(z')\cdot\nabla_n-\frac{b}{\eta(n)}\widetilde{S}(\mathbf{n},\Pi(z'))\right]\right]\right\}_+\delta(\mathbf{n}-\mathbf{n}')\ . \quad (10)$$

The derivative in the ordered exponential can be removed by using the identity

$$\left[\exp\left[-\int_0^z dz'(A(\mathbf{n})+\mathbf{Q}(z')\cdot\nabla_\mathbf{n})\right]\right]_+F(\mathbf{n})=\left\{\exp\left[-\int_0^z dz'\,A\left[\mathbf{n}-\int_{z'}^z dz''\,\mathbf{Q}(z'')\right]\right]\right\}F\left[\mathbf{n}-\int_0^z dz'\,\mathbf{Q}(z')\right]$$

(this identity can be proven by noticing that both sides of the equality satisfy the same differential equation with respect to z and the same initial condition). When $\mathbf{Q}=0$, the functional derivative term $\exp\int\phi\cdot(\delta/\delta\mathbf{Q})$ simply replaces $\mathbf{Q}$ everywhere by ϕ. Defining a new integration variable $\beta(z')=\mathbf{n}-\int_{z'}^z dz''\,\phi(z'')$, the path integral measure $[D\phi]$ becomes

$$\left[\mathrm{Det}\frac{\partial}{\partial z}\right]^2[D\beta]\delta(\mathbf{n}-\beta(z))\ ,$$

and the solution of the RT equation is

$$\widetilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z)=N^{-1}\int[D\Pi][D\beta]\delta(\mathbf{n}-\beta(z))\delta(\mathbf{n}'-\beta(0))\exp\left[i\int_0^z dz'\,\dot{\beta}(z')\cdot\Pi(z')\right]\exp\left[-\int_0^z dz'\,L(\mathbf{k},\beta(z'))\right]$$
$$\times\exp\left[\int_0^z dz'\frac{b}{\eta(n)}\widetilde{S}(\beta(z'),\Pi(z'))\right] \quad (11)$$

(the determinant $[\mathrm{Det}(\partial/\partial z)]^2$ was absorbed into the normalization constant N^{-1}). The constant N^{-1} is determined by the initial condition [Eq. (7)].

The path integral solution equation (11) is the general solution of the RT equation desired. This solution for the MTF is analogous in form to the path integral solution for the evolution operator in quantum mechanics; i.e., for a wave function $\Psi(q,t)$ which evolves in time according to

$$\Psi(q,t)=\int dq'\,K(q,t;q',0)\Psi(q',0)\ ,$$

the evolution operator K has the path integral representation

$$K(q_f,t;q_i,0)=\int[Dp][Dq]\delta(q(0)-q_i)\delta(q(t)-q_f)\times\exp\left[i\int_0^t dt'[\dot{q}p-\mathscr{H}(q,p)]\right]$$

where $\mathscr{H}(q,p)$ is the Hamiltonian. This parallel form suggests a dynamical interpretation of Eq. (11) in which β plays the role of a position variable, constrained within the unit disk, Π its conjugate momentum, and z the "time" parameter. Each path (β,Π) in phase space is included in the sum in Eq. (11) with a weight

$$\exp\left[-\int_0^z dz'\left[\frac{c-b\widetilde{S}(\beta,\Pi)}{\eta(\beta)}\right]\right]\ ,$$

which favors paths that closely follow the z axis when the phase function is dominated by forward scattering. Large deviations from the z axis are possible when the phase function allows large-angle scattering.

Also seen in Eq. (11) is the reason why the solution is limited to the forward-scattering region of $(\mathbf{r},\hat{\mathbf{n}})$ space $(z>0,\ \theta<\pi/2)$. For $z<0,\ \theta<\pi/2$ or $z>0,\ \theta>\pi/2$, the total attenuation term $\exp(-\int c/\eta)$ would diverge with increasing distance from the initial plane $z=0$.

Two approximations which lead to well-known solutions are considered in Secs. III and IV. The first (Sec. III) is a discrete scattering approximation which develops as a perturbation expansion in the number of scattering events, and the second (Sec. IV) is the small-angle approximation. In Sec. V a small-angle solution is found by considering the effect of absorption on the set of dominant phase-space paths. This solution is inequivalent to the small-angle solution of Sec. IV except in the $z\to 0$ limit.

III. DISCRETE SCATTERING APPROXIMATION

Consider the scattering coefficient b to be sufficiently small such that scattering is weak compared to absorption. In such a case the radiance is a sum of direct radiation, single-scattered radiation, doubly-scattered radiation, etc. This expansion is obtained from the general solution, Eq. (11), by a Taylor expansion of the scattering term:

$$\widetilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z)=N^{-1}\sum_{m=0}^{\infty}\frac{b^m}{m!}\int_0^z\prod_{i=1}^{m}dz_i\int[D\beta][D\Pi]\delta(\mathbf{n}-\beta(z))\delta(\mathbf{n}'-\beta(0))\prod_{i=1}^{m}\widetilde{S}(\beta(z_i),\Pi(z_i))$$
$$\times\exp\left[\int_0^z dz'[i\dot{\beta}\cdot\Pi-L(\mathbf{k},\beta)]\right]\ .$$

The first term is the direct radiation term:

$$\tilde{G}_0(\mathbf{k},\mathbf{n},\mathbf{n}',z)=\delta(\mathbf{n}-\mathbf{n}')\exp[-zL(\mathbf{k},\mathbf{n})] . \tag{12}$$

The second term is the single-scattered radiation:

$$\tilde{G}_1(\mathbf{k},\mathbf{n},\mathbf{n}',z) =bS(\mathbf{n},\mathbf{n}')\frac{\exp[-zL(\mathbf{k},\mathbf{n})]-\exp[-zL(\mathbf{k},\mathbf{n}')]}{L(\mathbf{k},\mathbf{n})-L(\mathbf{k},\mathbf{n}')} . \tag{13}$$

The mth term describes the radiation after m scattering events.

IV. SMALL-ANGLE SOLUTION I

We now consider a phase function which is sharply forward peaked and allow only small angles. Then $\tilde{S}(\beta,\Pi)\to\tilde{S}(\Pi)$ and $L(k,\beta)\to i\mathbf{k}\cdot\beta+c$. If we set

$$\beta(z')=\mathbf{n}'+(\mathbf{n}-\mathbf{n}')(z'/z)+\gamma(z') ,$$

then the path integration over γ can be evaluated to give

$$\prod_{z'}\delta(\dot{\Pi}(z')-\mathbf{k}) .$$

Setting $\Pi(z')=\mathbf{k}z'+\mathbf{p}$,

$$\tilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z)=\exp[-z(c+i\mathbf{k}\cdot\mathbf{n}')] \times\int\frac{d^2p}{(2\pi)^2}\exp[i\mathbf{p}\cdot(\mathbf{n}-\mathbf{n}')] \times\exp\left[b\int_0^z dz'\,\tilde{S}(\mathbf{k}z'+\mathbf{p})\right] . \tag{14}$$

This is a well-established result for small-angle scattering. From the dynamical point of view, this solution consists of a set of phase-space paths $(\mathbf{n},\mathbf{p})$ whose weighting is decided entirely by the phase function. Paths which have a high "cost," i.e., for which the attenuation $-b\int\tilde{S}$ is large, do not contribute to the MTF. This ignores the role played by the absorption, which excludes long gently curving paths which may be allowed by the phase function.

A particular small-angle solution is obtained by choosing the commonly used phase function

$$\tilde{S}(\mathbf{p})=1-\frac{\mu^2}{2}p^2 ,$$

where $\mu^2=\int d^2n\,S(n)n^2\ll 1$ and μ is the rms scattering angle per scattering event. For example, this phase function can be obtained from a Fourier transformation of the forward-peaked Arnush phase function

$$S(\theta)=\frac{1}{2\pi\theta\mu}\exp(-\theta/\mu)$$

in the $\mu\ll 1$ limit. In this case,

$$\tilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z) =\frac{\exp(-az)}{2\pi\mu bz}\exp\left[-i\frac{z}{2}\mathbf{k}\cdot(\mathbf{n}+\mathbf{n}') -\frac{(\mathbf{n}-\mathbf{n}')^2}{2\mu bz}-\frac{\mu b}{24}k^2z^3\right] . \tag{15}$$

This solution [Eq. (14)] can also be obtained directly from the RT equation by interpreting Eq. (5) as an invertible Fourier transformation of a forward-peaked phase function in the small-angle limit. Stotts[5] has argued that the diffusive solution [Eq. (15)] is applicable to radiance distributions which have only low-order spatial and angular Fourier components. To handle higher-order components, multiple scattering must be included. To study the effects of higher-order components, a small-angle solution is constructed in Sec. V which includes the effect of absorption in selecting the paths which have the largest contribution to the MTF. For comparison of that solution with the solution (15), an initial radiance distribution consisting of a point beam is considered, for which

$$\tilde{I}_0(\mathbf{k},\mathbf{n})=\delta(\mathbf{n}) .$$

The radiance obtained using Eq. (15) is

$$I(\mathbf{x},\mathbf{n},z) =\frac{3\exp(-az)}{(\pi\mu bz^2)^2}\exp\left[-\frac{n^2}{2\mu bz}-\frac{6}{\mu bz^3}\left[\mathbf{x}-\frac{z}{2}\mathbf{n}\right]^2\right] . \tag{16}$$

V. SMALL-ANGLE SOLUTION II

The small-angle solution, Eq. (15), could be constructed because the small-angle restriction and choice of phase function left only terms up to quadratic in the exponent of Eq. (11), with the only quadratic term coming from the phase function. A reexamination of Eq. (11) in the small-angle limit shows that an additional quadratic term can be contributed by the absorption term. After evaluating the Π integration, Eq. (11) becomes

$$\tilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z)=N^{-1}\exp(-az)\int[D\beta]\delta(\beta(0)-\mathbf{n}')\delta(\beta(z)-\mathbf{n})\exp\left[-\int_0^z dz'\left[i\mathbf{k}\cdot\beta(z')+\frac{1}{2\mu b}(\dot{\beta}^2+\beta^2/l^2)\right]\right] , \tag{17}$$

where

$$l^2=(\mu ab)^{-1} \tag{18}$$

is a "diffusive path length." Its significance is as follows: for $z\ll l$, the phase function controls the choice of paths which dominate in Eq. (17) because the length of the possible paths do not vary greatly and are less than l. Consequently the angular rate of change is minimized, and as the radiance travels from its initial angle $\mathbf{n}'$ to the final angle $\mathbf{n}$, it undergoes as many scattering events as possible, each through the smallest angular deviation possible. At larger distances, $z\gg l$, the diffusive solution is modified by the effect of absorption. The set of dominant paths now includes ones which have relatively large-angle scattering events. These paths are chosen by a balance between the tendency of absorption to minimize path length, narrowing the distribution, and the phase function to maximally diffuse and broaden the radiance. In such a balance the set of dominant paths can include ones which have a few relatively large-angle scatters compared to the diffusive solution, the large cost of which is counterbalanced by a reduction in the absorption. This is the type of path which contributes to a multiple-scattering solution. The length scale l defines the transition between the region of maximal diffusion and the nondiffusive region that includes these multiple-scattering-like effects. For the point beam, the radiance has a narrower distribution than the diffusive solution both spatially and angularly, and the spatial peak of the distribution is closer to the beam axis.

The evaluation of Eq. (17) is carried out in the Appendix. To distinguish it from the small-angle (SA) solution of Eq. (15), it will be called the small-angle with absorption (SAA) solution. Explicitly, it is

$$\tilde{G}_{\rm SAA}(\mathbf{k},\mathbf{n},\mathbf{n}',z) = \overline{G}(z/l)\exp[-az - i\mathbf{k}\cdot(\mathbf{n}+\mathbf{n}')lR(z/l)] \times \exp[-A(z/l,\mathbf{n},\mathbf{n}') - \tfrac{1}{2}\mu b k^2 l^3 h(z/l)] , \tag{19}$$

where

$$\overline{G}(x) = [2\pi\mu b l \sinh(x)]^{-1} , \tag{20}$$

$$R(x) = \frac{\cosh(x)-1}{\sinh(x)} , \tag{21}$$

$$A(x,\mathbf{n},\mathbf{n}') = [2\mu b l \sinh^2(x)]^{-1} \times \left[\frac{n^2+(n')^2}{2}\sinh(2x) - 2\mathbf{n}\cdot\mathbf{n}'\sinh(x) \right] , \tag{22}$$

$$h(x) = \frac{x\sinh(x)+2[1-\cosh(x)]}{\sinh(x)} . \tag{23}$$

The two asymptotic regimes are the diffusive regime $z \ll l$, where

$$R \sim z/2l ,$$
$$h \sim \tfrac{1}{12}(z/l)^3 ,$$
$$A \sim (2\mu b z)^{-1}(\mathbf{n}-\mathbf{n}')^2 ,$$
$$\overline{G} \sim (2\pi\mu b z)^{-1} ,$$

which is the SA solution of Eq. (15); and the nondiffusive regime $z \gg l$, where

$$R \sim 1 ,$$
$$h \sim z/l ,$$
$$A \sim [n^2+(n')^2]/2\mu b l ,$$
$$\overline{G} \sim (\pi\mu b l)^{-1}\exp(-z/l) .$$

Notice that $\mathbf{n}$ and $\mathbf{n}'$ have decoupled in A in the nondiffusive region, giving the prediction that a spatially invariant radiance at $z=0$ will have an asymptotic Gaussian distribution for z sufficiently large, with an angular width depending on only the scattering and absorption properties of the medium.

To compare with the SA solution for the point beam, the corresponding SAA solution is

$$I_{\rm SAA}(\mathbf{x},\mathbf{n},z) = [(2\pi\mu b l^2)^2 h(z/l)\sinh(z/l)]^{-1} \times \exp\{-[az + M(\mathbf{x},\mathbf{n},z)]\} , \tag{24}$$

with

$$M(\mathbf{x},\mathbf{n},z) = n^2\sinh(2z/l)/4\mu b l \sinh^2(z/l) + \tfrac{1}{2}[\mathbf{x}-\mathbf{n}lR(z/l)]^2 . \tag{25}$$

The distribution is spatially and angularly Gaussian in both cases (16) and (24). Four measures of the scattering properties of either solution are the spatial and angular peaks and widths. These are SA:

$$\mathbf{x}_p = \mathbf{n}z/2, \quad \sigma_x^2 = \mu b z^3/24 , \tag{26a}$$

$$\mathbf{n}_p = \tfrac{3}{2}\mathbf{x}/z, \quad \sigma_n^2 = \mu b z/4 ; \tag{26b}$$

and SAA ($z \gg l$):

$$\mathbf{x}_p = \mathbf{n}l, \quad \sigma_x^2 = \mu b z l^2/2 , \tag{27a}$$

$$\mathbf{n}_p = 2\mathbf{x}/z, \quad \sigma_n^2 = \mu b l . \tag{27b}$$

In the large-z limit, the SAA solution predicts a radiance which is concentrated into a narrower beam than the SA solution. Because $\mathbf{n}_p$ is larger in magnitude in the SAA solution, the radiance arriving at any given point $\mathbf{x}$ appears to come from a point on the beam axis closer to $(\mathbf{x},z)$ than for the SA solution. This is consistent with the mechanism discussed earlier that increases the importance of large-angle scattering events in the SAA solution over the SA solution.

APPENDIX: THE SAA SOLUTION

The path integral to be evaluated is

$$\tilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z) = N^{-1}\exp(-az)\int[D\beta]\delta(\beta(0)-\mathbf{n}')\delta(\beta(z)-\mathbf{n})\exp\left[-\int_0^z dz'\left[i\mathbf{k}\cdot\beta(z') + \frac{1}{2\mu b}(\dot{\beta}^2+\beta^2/l^2)\right]\right] . \tag{A1}$$

The integration variable $\beta(z')$ is expanded as

$$\beta(z') = \mathbf{y}(z') + \gamma(z') ,$$

where $\mathbf{y}$ satisfies

$$\left[-\frac{\partial^2}{\partial(z')^2}+1/l^2\right]\mathbf{y}(z')=0\ , \tag{A2}$$

and the boundary conditions

$$\mathbf{y}(0)\mathbf{n}',\ \ \mathbf{y}(z)=\mathbf{n}\ . \tag{A3}$$

The solution for $\mathbf{y}$ is

$$\mathbf{y}(z')=\left[\mathbf{n}\sinh(z'/l)+\mathbf{n}'\sinh\left[\frac{z-z'}{l}\right]\right]\Big/\sinh(z/l)\ . \tag{A4}$$

and the integral is now

$$\tilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z)=N^{-1}\exp\left\{-\left[az+\int_0^z dz'\left[i\mathbf{k}\cdot\mathbf{y}(z')+\frac{1}{2\mu b}(\dot{y}^2+y^2/l^2)\right]\right]\right\}$$

$$\times\int[D\gamma]\delta(\gamma(0))\delta(\gamma(z))\exp\left[-i\int_0^z dz'\,\mathbf{k}\cdot\gamma(z')\right]\exp\left[-\frac{1}{2\mu b}\int_0^z dz'(\dot{\gamma}^2+\gamma^2/l^2)\right]. \tag{A5}$$

The evaluation of this Gaussian path integral gives [using Eq. (7) for the initial condition]

$$\tilde{G}(\mathbf{k},\mathbf{n},\mathbf{n}',z)=(\mathrm{Det}\Delta^{-1})\exp\left[-\left[az+i\mathbf{k}\cdot\mathbf{R}(z)+\frac{\mu b}{2}k^2H(z)+F(z)\right]\right], \tag{A6}$$

where

$$\begin{aligned}
\mathbf{R}(z)&=\int_0^z dz'\,\mathbf{y}(z')\ ,\\
H(z)&=\int_0^z dz'\int_0^z dz''\,\Delta^{-1}(z',z'')\ ,\\
\Delta&=-\frac{\partial^2}{\partial z^2}+1/l^2\ ,\\
F(z)&=\int_0^z dz'(\dot{y}^2+y^2/l^2)\ .
\end{aligned} \tag{A7}$$

The inverse of the operator Δ is

$$\Delta^{-1}(z',z'')=\frac{l}{\sinh(z/l)}\left[\Theta(z'-z'')\sinh(z''/l)\sinh\left[\frac{z-z'}{l}\right]+\Theta(z''-z')\sinh(z'/l)\sinh\left[\frac{z-z''}{l}\right]\right], \tag{A8}$$

where Θ is the Heaviside step function. The determinant is

$$\mathrm{Det}\Delta^{-1}=\int[D\beta]\delta(\beta(0))\delta(\beta(z))\times\exp\left[-\frac{1}{2\mu b}\int_0^z dz'(\dot{\beta}^2+\beta^2/l^2)\right]. \tag{A9}$$

A similar determinant has been evaluated[9] as

$$\mathrm{Det}\bar{\Delta}^{-1}=\int[D\beta]\delta(\beta(0))\delta(\beta(z))\times\exp\left[\frac{i}{2}\int_0^z dz'(\dot{\beta}^2-\beta^2/l^2)\right]=[il\sin(z/l)]^{-1}\ . \tag{A10}$$

The path integral in (A10) becomes (A9) by the analytic continuation $z\to -iz$, which gives

$$\mathrm{Det}\Delta^{-1}=[l\sinh(z/l)]^{-1}\ . \tag{A11}$$

Combining the pieces (A4)—(A11), the result is Eq. (19).

The "mean path" of Eq. (A4) is the dominant path taken by the radiation. Because of the character of the sinh function, the approach of the radiation to the final angle $\mathbf{n}$ is slow for much of the path, accelerating as the radiation approaches the observation point. This leads to the interpretation that large-angle scattering events are more important in this SAA solution than in the SA solution.

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