The Map of a Sphere to and from the Image Plane

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1 Summary

It is obvious that the image plane silhouette of a 3D sphere is not always a circular disk, and is also not an ellipse. We will demonstrate that the image plane silhouette of a sphere is a quadratic function of position on the image plane, and that the shape is easily described as a slice through 3D ellipsoid. An explicit formula is given for the shape. The key quantities that control the shape are the position of the center of the sphere in the image plane, and the ratio of the sphere radius to its distance from the camera.

This analysis gives us three important algorithms:

1. an algorithm for testing whether an image plane point is inside the disk or not.
2. an algorithm for computing the point on the disk edge that lies on the line between an inside point and an outside point.
3. an algorithm for mapping a point on the image plane inside the disk to the two corresponding points on the 3D sphere.

The first two results provide a method for doing clipping in the image plane, and the third one allows us to track back to points on the sphere from the image plane, without doing an explicit ray trace.

In the special case that the sphere is aligned on the axis of the camera view direction, the image plane shape is a circular disk, with radius $\epsilon_0/\sqrt{1-\epsilon_0^2}$, where the dimensionless variable q

$$\epsilon_0 = \frac{R}{|\vec{\mathbf{r}}_S - \vec{\mathbf{r}}_C|}\ , \tag{1}$$

is less than one. Here, R is the radius of the 3D sphere, $\vec{\mathbf{r}}_S$ is the 3D position of the sphere, and $\vec{\mathbf{r}}_C$ is the 3D position of the camera.

2 Imaging a Sphere

The basic equation for imaging maps points $\vec{\mathbf{r}}$ in 3D space to points $\vec{\mathbf{x}}$ on the image plane. Using the camera position $\vec{\mathbf{r}}_C$ and the camera pointing direction $\hat{\mathbf{n}}$, the imaging equation for the perfect camera is

$$\vec{\mathbf{x}} = \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}_C}{\hat{\mathbf{n}}\cdot(\vec{\mathbf{r}} - \vec{\mathbf{r}}_C)} - \hat{\mathbf{n}} \tag{2}$$

It is easy to verify that, although $\vec{\mathbf{x}}$ is written as a point in 3D space, the locus of such points at $\vec{\mathbf{r}}$ varies all lie on a plane that has $\hat{\mathbf{n}}$ as its normal. Also, the units of $\vec{\mathbf{x}}$ are dimensionless "tangent" units.

The 3D sphere we imagine to be centered at the point $\vec{\mathbf{r}}_0$ sufficiently far away that the sphere is entirely in front of the image plane (although it may be out of the FOV of a particular camera). With a radius R, the points on the surface of the sphere have the representation

$$\vec{\mathbf{r}}_0 + R\hat{\phi}\ , \tag{3}$$

where the unit vector $\hat{\phi}$ points in all directions. The imaging equation for the points on the surface is now

$$\vec{\mathbf{x}}(\hat{\phi}) = \frac{\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C + R\hat{\phi}}{\hat{\mathbf{n}} \cdot (\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C + R\hat{\phi})} - \hat{\mathbf{n}} \tag{4}$$

The basic question of interest can now be phrased this way: Is the set of points $\{\vec{\mathbf{x}}(\hat{\phi}) | \hat{\phi} \in S^2\}$ on the image plane in the shape of a circular disk? If so, what is the radius and center of the disk?

3 Mapping a Sphere to the Image Plane

The center point of the sphere $\vec{\mathbf{r}}_0$ maps to the position

$$\vec{\mathbf{x}}_0 = \frac{\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C}{\hat{\mathbf{n}} \cdot (\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C)} - \hat{\mathbf{n}} \tag{5}$$

in the image plane. The points $\vec{\mathbf{x}}(\hat{\phi})$ have the form $\vec{\mathbf{x}}(\hat{\phi}) = \vec{\mathbf{x}}_0 + \delta\vec{\mathbf{x}}(\hat{\phi})$, where

$$\delta\vec{\mathbf{x}}(\hat{\phi}) = \frac{\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C + R\hat{\phi}}{\hat{\mathbf{n}} \cdot (\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C + R\hat{\phi})} - \frac{\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C}{\hat{\mathbf{n}} \cdot (\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C)} \tag{6}$$

A little algebraic manipulation reveals that this has the form

$$\delta\vec{\mathbf{x}}(\hat{\phi}) = \epsilon \frac{\mathbf{M} \cdot \hat{\phi}}{1 + \epsilon \hat{\mathbf{n}} \cdot \hat{\phi}} \tag{7}$$

where $\epsilon = R/\hat{\mathbf{n}} \cdot (\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C) < 1$. The matrix $\mathbf{M}$ is a projection operator with a number of interesting properties. In detail, it has the form

$$\mathbf{M} = \mathbf{1} - \frac{\hat{\mathbf{r}}\hat{\mathbf{n}}}{\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}} \tag{8}$$

with $\hat{\mathbf{r}} = (\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C)/|\vec{\mathbf{r}}_0 - \vec{\mathbf{r}}_C|$. Here are the interesting properties:

- $\mathbf{M}$ is idempotent: $\mathbf{M} \cdot \mathbf{M} = \mathbf{M}$.
- $\mathbf{M} \cdot \hat{\mathbf{r}} = 0$: all points of the sphere along the line $\vec{\mathbf{r}}_0 + t\hat{\mathbf{r}}$ map to the point $\vec{\mathbf{x}}_0$ on the image plane.
- $\hat{\mathbf{n}} \cdot \mathbf{M} = 0$: the points $\delta\vec{\mathbf{x}}(\hat{\phi})$ are still on the image plane, as should be expected.
- The commutator $[\mathbf{M}, \mathbf{M}^T] \neq 0$, but

$$[\mathbf{M}, \mathbf{M}^T] = \frac{\hat{\mathbf{r}}\hat{\mathbf{r}} - \hat{\mathbf{n}}\hat{\mathbf{n}}}{(\hat{\mathbf{r}} \cdot \hat{\mathbf{n}})^2} \tag{9}$$

  is symmetric.

4 Mapping from the Image Plane Disk to the Sphere

With a little more manipulation, equation 7 provides an algorithm for mapping a point $\delta\vec{\mathbf{x}}$ on the image plane back to the two corresponding points on the 3D sphere. That algorithm is constructed in this section. Some properties of the algorithm will serve in the other algorithms of interest.

We will make progress by defining a constant α to produce the coupled equations

$$1 + \epsilon \hat{\mathbf{n}} \cdot \hat{\phi} = \epsilon \alpha \tag{10}$$

$$\mathbf{M} \cdot \hat{\phi} = \alpha \delta\vec{\mathbf{x}} \, . \tag{11}$$

From the second of these equations, and from the definition of $\mathbf{M}$, we obtain

$$\hat{\phi} = \alpha \delta\vec{\mathbf{x}} + \frac{\hat{\mathbf{r}}(\hat{\mathbf{n}} \cdot \hat{\phi})}{\hat{\mathbf{n}} \cdot \hat{\mathbf{r}}} \, . \tag{12}$$

From equation 10 $\hat{\mathbf{n}} \cdot \hat{\phi} = \alpha - 1/\epsilon$, so the explicit expression for $\hat{\phi}$ is

$$\hat{\phi} = \alpha \left(\delta\vec{\mathbf{x}} + \vec{\mathbf{z}} \right) - \frac{1}{\epsilon} \vec{\mathbf{z}} \tag{13}$$

with

$$\vec{\mathbf{z}} = \frac{\hat{\mathbf{r}}}{\hat{\mathbf{r}} \cdot \hat{\mathbf{n}}} \, . \tag{14}$$

While this equation for $\hat{\phi}$ is explicit, we still need to know the value of α. Also, since two points on the sphere project to the point $\delta\vec{\mathbf{x}}$, we can expect two values of α. That is indeed the case, because the unit norm of $\hat{\phi}$ generates the following quadratic equation for α:

$$\alpha^2 \left| \delta\vec{\mathbf{x}} + \vec{\mathbf{z}} \right|^2 - 2\frac{\alpha}{\epsilon} \vec{\mathbf{z}} \cdot \left(\delta\vec{\mathbf{x}} + \vec{\mathbf{z}} \right) + \frac{z^2}{\epsilon^2} - 1 = 0 \, . \tag{15}$$

The two solutions are

$$\alpha_{\pm} = \frac{\vec{\mathbf{z}} \cdot \left(\delta\vec{\mathbf{x}} + \vec{\mathbf{z}} \right) \pm \left\{ \left(\vec{\mathbf{z}} \cdot \left(\delta\vec{\mathbf{x}} + \vec{\mathbf{z}} \right) \right)^2 - \left(z^2 - \epsilon^2 \right) \left| \delta\vec{\mathbf{x}} + \vec{\mathbf{z}} \right|^2 \right\}^{1/2}}{\epsilon \left| \delta\vec{\mathbf{x}} + \vec{\mathbf{z}} \right|^2} \tag{16}$$

Equations 13 through 16 constitute a complete solution for mapping from the image plane back to points on the sphere.

5 The Silhouette as a Slice of a 3D Ellipsoid

Note that for α to exist as a real number, the quantity inside the square root must be positive. Points in the image plane which have a negative value for the radical do not map to the 3D sphere. So the silhouette of the disk is the collection of points which make the radical zero. Setting it to zero and rearranging terms a little, we can write the equation for the silhouette as

$$\xi(\delta\vec{\mathbf{x}}) \equiv \left(\delta\vec{\mathbf{x}} + \vec{\mathbf{z}} \right) \cdot \mathbf{Q} \cdot \left(\delta\vec{\mathbf{x}} + \vec{\mathbf{z}} \right) = 0 \, , \tag{17}$$

with the matrix $\mathbf{Q}$ being

$$\mathbf{Q} = \mathbf{1} - \frac{\vec{\mathbf{z}}\vec{\mathbf{z}}}{z^2 - \epsilon^2} \tag{18}$$

Because $\vec{\mathbf{z}}$ and $\mathbf{Q}$ are functions only of $\hat{\mathbf{r}}$ and $\hat{\mathbf{n}}$, $\xi(\delta\vec{\mathbf{x}})$ is a quadratic function of $\delta\vec{\mathbf{x}}$ that corresponds to a 3D ellipsoid centered in 3D space at $-\vec{\mathbf{z}}$. Since the vectors $\delta\vec{\mathbf{x}}$ always lie in the image plane, the silhouette is the shape obtained by slicing the 3D ellipsoid with a plane co-located and co-oriented with the image plane.

6 Testing for Inside/Outside Status

Because we can characterize the silhouette as satisfying equation 17, we can build an algorithm to test for whether a point in the image plane in inside the disk or not. Defining the test function

$$T(\delta\vec{\mathbf{x}}) = |\delta\vec{\mathbf{x}} + \vec{\mathbf{z}}|^2 - \frac{|\vec{\mathbf{z}} \cdot (\delta\vec{\mathbf{x}} + \vec{\mathbf{z}})|^2}{z^2 - \epsilon^2} \tag{19}$$

the test for inside/outside conditions is

$$\begin{aligned} T(\delta\vec{\mathbf{x}}) &\quad > 0 \quad \text{Outside} \\ T(\delta\vec{\mathbf{x}}) &\quad \leq 0 \quad \text{Inside} \end{aligned}$$

As an interesting special case, if the sphere is centers on the camera axis, $\hat{\mathbf{r}} = \hat{\mathbf{n}}$, then the test function T reduces to

$$T_{\text{special}} = |\delta\vec{\mathbf{x}}|^2 - \frac{\epsilon^2}{1 - \epsilon^2} \tag{20}$$

which is the test for a circular disk.

7 Intersection of a Line and the Silhouette

For our last trick, we consider the following problem, related to clipping polygons with the disk: Suppose the image plane point $\delta\vec{\mathbf{x}}_0$ lies in the disk (after $T(\delta\vec{\mathbf{x}}_0)$ tested negative), and $\delta\vec{\mathbf{x}}_1$ lies outside the disk. What is the point along the line segment between these two points that lies on the edge of the disk?

To solve this, the points on the line segment are parameterized as

$$\delta\vec{\mathbf{x}}(t) = \delta\vec{\mathbf{x}}_0 + t\delta^2\vec{\mathbf{x}} \ , \tag{21}$$

with $\delta^2\vec{\mathbf{x}} = \delta\vec{\mathbf{x}}_1 - \delta\vec{\mathbf{x}}_0$. The points on the line segment are the collection for which $0 \leq t \leq 1$. Anything outside that range is not in the line segment. Putting this expression for $\delta\vec{\mathbf{x}}(t)$ into ξ, the result is a quadratic equation for t:

$$At^2 + Bt + C = 0 \ , \tag{22}$$

with

$$A = \delta^2\vec{\mathbf{x}} \cdot \mathbf{Q} \cdot \delta^2\vec{\mathbf{x}} \tag{23}$$

$$B = 2\delta^2\vec{\mathbf{x}} \cdot \mathbf{Q} \cdot (\delta\vec{\mathbf{x}}_0 + \vec{\mathbf{z}}) \tag{24}$$

$$C = (\delta\vec{\mathbf{x}}_0 + \vec{\mathbf{z}}) \cdot \mathbf{Q} \cdot (\delta\vec{\mathbf{x}}_0 + \vec{\mathbf{z}}) \tag{25}$$

Because $\delta\vec{\mathbf{x}}_0$ is inside the disk, we know that one of the solutions of this quadratic equation is position and one is negative. We only need the positive solution, which is

$$t = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad . \tag{26}$$

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