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Measures of temporal pulse stretching

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Abstract

Temporal pulse stretching is a consequence of the multiple scatter by ocean water of a laser pulse. Although the physical process behind pulse stretching is intuitively clear, there is no widely held quantitative definition of it. Here temporal pulse stretching is defined in terms of temporal moments of the radiance at a fixed position and orientation with respect to the initial pulse axis. This definition has been chosen because it is directly measurable from the waveform output of a radiometer. The first temporal moment is a measure of the apparent delay of the pulse, and the variance from the second moment describes the increasing width. Using a WKB approach, an expression is obtained for the first two temporal moments for waveforms measured at positions along the initial pulse axis. Quantitative predictions of the temporal delay and width are made for a pulse which is initially a collimated point. To within an error of no more than 12%, the delay and width are proportional. Stretching effects on waveforms are shown graphically in plots at various distances from the source.

1 Introduction

Although laser pulses propagating in the ocean suffer the same absorption and scattering processes as constant light sources, the radiative properties of pulses are significantly different. This is because the radiance distribution evolves in time in any reference frame due to multiple scattering. The overall multiple scattering effect on the pulse is a broadening (or stretching) of the pulse volume and a widening of its angular divergence. However, scattered photons lag behind the original pulse, and never move ahead of it, so that broadening also skews the volumetric structure of the radiance field toward the back end of the pulse as it evolves. The purpose of the work in this paper is to quantify both the skewing and stretching in a set of numerical and analytical calculations.

Although the intuitive picture of multiple scatter stretching of the pulse is clear, there is no widely held quantitative definition of a measure of the phenomenon. One of the first attempts

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to describe stretching in the ocean optics context was by Stotts[1], who characterized stretching in terms of the mean excess path length photons travel when they arrive at a plane some distance from their initial point. The expression he obtained agreed reasonably with the numerical results of McLean et al[2] using a Monte Carlo algorithm to follow individual rays. This approach combines together all photons arriving at any point at a fixed depth plane at any angle and any time, and so is not amenable to time-series measurements from a single radiometer.

A convenient description we adopt here is in terms of the impact on temporal waveforms output by a radiometer looking at the oncoming pulse. Temporal moments of the waveform quantify the evolution of the pulse as a function of the radiometer distance from the initial point, and are derivable from measurements. However, temporal moments are also dependent on the initial pulse distribution, so that the values of the moments must be considered in the context of the particular initial distribution. For the purposes of this work, an initially collimated point pulse is used to discuss stretching in order too avoid the complication of pulse size effects.

Dolin[3] and Ishimaru[4] examined the temporal structure of pulses using the small-angle approximation to solve the time-dependent radiative transfer equation. Their expressions for the first two moments agree in broad character with those found in section 3 below, although the results are much larger and predict too much broadening. The reason for the disagreement is discussed below.

In section 2 temporal moments are constructed from the distribution of scattered radiance. This is accomplished in the framework of the path integral solution of the time-dependent radiative transfer equation. After separating the scattered from the unscattered components of the solution, expressions for temporal moments are obtained from a WKB approximation of the path integral. The first two moments, describing the time delay and width of the pulse, are discussed in detail in section 3 where a relationship between the two is found. In section 4 pulse stretching is further illustrated in the waveforms predicted by the WKB approximation. Although the magnitude of stretching effects revealed in the first two moments is not large, it is clear from the long tails in the waveforms that time delay and width do not tell the whole story of pulse stretching.

2 Temporal moments from waveforms

We describe an evolving pulse in terms of its radiance distribution $L(s, \vec{x}, \hat{n})$. The time variable s is the time multiplied by the speed of light in water, and so has units of length. Since the pulse originates as a collimated point, the initial distribution has the simple form

$$L(0, \vec{x}, \hat{n}) = L_0 \, \delta(\vec{x}) \, \delta(\hat{n} - \hat{n}_0) \ , \tag{1}$$

where $\hat{n}_0$ is the collimation direction.

Suppose an idealized radiometer at position $\vec{x}$ receiving radiance from a solid angle centered on direction $\hat{n}$ has a very small field of view. The time series of received irradiance contains two components. The first is the unscattered pulse which has been attenuated by the amount exp($-cs$), where c is the extinction coefficient. For a point pulse as used here, this component resides at the front of the light cone and passes by the radiometer first. The second component is the scattered portion of the pulse, and is the portion of interest for pulse stretching. Writing the pulse in terms of these two components as

$$L(s,\vec{x},\hat{n}) = L_{Unscattered}(s,\vec{x},\hat{n}) + L_{Scattered}(s,\vec{x},\hat{n}) \; , \tag{2}$$

the scattered portion has temporal moments

$$\langle s^n \rangle (\vec{x},\hat{n}) = \frac{\int_0^\infty ds \; s^n L_{Scattered}(s,\vec{x},\hat{n})}{\int_0^\infty ds \; L_{Scattered}(s,\vec{x},\hat{n})} \; . \tag{3}$$

We will confine ourselves to examining moments along the beam axis in the collimation direction, so that from this point on we choose

$$\vec{x} = x\,\hat{n}_0 \tag{4}$$

$$\hat{n} = \hat{n}_0 \tag{5}$$

From the formulation of radiative transfer shown in section 2.1, the denominator in equation 3 can act as a generating function for temporal moments using the absorption coefficient a:

$$\langle s^n \rangle (x) = \frac{(-1)^n}{\mathcal{Z}(a,x)} \frac{\partial^n}{\partial a^n} \mathcal{Z}(a,x) \; , \tag{6}$$

where $\mathcal{Z}$ is the generating function

$$\mathcal{Z}(a,x) = \int_0^\infty ds \; L_{Scattered}(s, x\,\hat{n}_0, \hat{n}_0) \; . \tag{7}$$

The dependence on the absorption coefficient in $\mathcal{Z}$ is implicit. An explicit expression is constructed in section 2.2 for $\mathcal{Z}$.

In the absence of scattering, a point pulse has the moments

$$\langle s^n \rangle (x) = x^n \; . \tag{8}$$

When scattering is present the delay Δs is defined as the excess mean time after the earliest moment of arrival:

$$\Delta s = \langle s \rangle - x \; . \tag{9}$$

We will define the width of the pulse σ_s in terms of the temporal variance:

$$\begin{aligned} \sigma_s(x) &= \left\{ \langle s^2 \rangle - \langle s \rangle^2 \right\}^{1/2} \\ &= \left\{ \frac{1}{\mathcal{Z}(a,x)} \frac{\partial^2}{\partial a^2} \mathcal{Z}(a,x) - \left(\frac{1}{\mathcal{Z}(a,x)} \frac{\partial}{\partial a} \mathcal{Z}(a,x) \right)^2 \right\}^{1/2} \; . \end{aligned} \tag{10}$$

2.1 Formulation of radiative transfer

In order to obtain the delay and width, the generating function will be evaluated below using an approximate analytic approach. We begin by providing an exact explicit expression for the scattered radiance from a collimated point beam. The full radiance distribution has been written previously in terms of a path integral[5]

$$\begin{aligned} L(s,\vec{x},\hat{n}) &= N\int[d\hat{\beta}][dp]\ \delta\left(\hat{\beta}(0)-\hat{n}_0\right)\ \delta\left(\hat{\beta}(s)-\hat{n}\right)\ \delta\left(\vec{x}-\int_0^s ds'\ \hat{\beta}(s')\right) \\ &\times\ \exp(-cs)\exp\left(b\int_0^s ds'\ Z(\vec{p}(s'))\right)\exp\left\{i\int_0^s ds'\ \vec{p}(s')\cdot\frac{d\hat{\beta}(s')}{ds'}\right\}\ . \end{aligned} \tag{11}$$

In this expression, the function $\hat{\beta}(s')$ is a unit vector that is tangent to an arbitrary path photons can travel from the origin to the position $\vec{x}$, initially in the direction $\hat{n}_0$ and ending in the direction $\hat{n}$. The measure notation $[d\hat{\beta}]$ integrates over all possible paths satisfying these boundary conditions. All paths have arclength s. Only those paths for which $x \leq s$ are included, which is insured by the delta function constraint

$$\vec{x}=\int_0^s ds'\ \hat{\beta}(s')\ . \tag{12}$$

This constraint also insures that photons travel at the physically correct speed of light, and so is very important in calculating temporal moments. When approximations to this path integral are evaluated, as is done in section 2.2, care must be taken to make sure that the unit vector character of $\hat{\beta}$ is not violated. In the small-angle approximation for example, direction vectors are replaced by expressions which simplify the ensuing algebra, but also violate the physical speed of light requirement. Problems such as calculating the solar distribution are long-time limits, and so the small-angle approximation may be acceptable since photon arrival times are not important. For the calculation of temporal moments as we wish to do here, appropriate approximations have been found which do not require the small-angle approximation and which do not violate the constant speed of light.

The phase function enters into this solution through its Fourier transform Z, and depends only on the magnitude of the Fourier transform variable $\vec{p}$:

$$Z(\vec{p})=2\pi\int_0^2 dq\ q\ P\left(1-\frac{q^2}{2}\right)J_0(pq)\ , \tag{13}$$

where $P(\cos\theta)$ is the phase function and J_0 is a Bessel function. By definition $Z(0)=1$, and as $p\to\infty$, $Z\to 0$.

In this small angular scale limit, the path integral does not go to zero. In order to approximate a path integral with an argument that goes to zero at infinity, we add and subtract 1 as in

$$\exp\left(b\int_0^s ds'\ Z(\vec{p}(s'))\right)\to\left[\exp\left(b\int_0^s ds'\ Z(\vec{p}(s'))\right)-1\right]+1\ . \tag{14}$$

The term in brackets is goes to zero as $p \to \infty$, and the path integrals can be evaluated exactly with the right hand 1 in place, to give

$$\begin{aligned} L(s,\vec{x},\hat{n}) &= \exp(-cs)\ L(s=0,\vec{x}-s\ \hat{n},\hat{n}) \\ &+ N\int [d\hat{\beta}][dp]\ \delta\left(\hat{\beta}(0)-\hat{n}_0\right)\ \delta\left(\hat{\beta}(s)-\hat{n}\right)\ \delta\left(\vec{x}-\int_0^s ds'\ \hat{\beta}(s')\right) \\ &\times\ \exp(-cs)\left[\exp\left(b\int_0^s ds'\ Z(\vec{p}(s'))\right)-1\right]\exp\left\{i\int_0^s ds'\ \vec{p}(s')\cdot\frac{d\hat{\beta}(s')}{ds'}\right\}\ . \end{aligned} \tag{15}$$

The first term is the unscattered radiance distribution, and the remaining path integral term is the scattered radiance we will use in the temporal moments.

In this solution for the radiance distribution, the absorption coefficient enters into the expression as an overall multiplication by $\exp(-as)$. This justifies the definition given above for the generating function of temporal moments.

2.2 WKB approximation

The approximate evaluation of the path integral in equation 15 proceeds by first expanding Z for small p as

$$Z(\vec{p}) \approx 1 - \frac{\langle\theta^2\rangle}{2}p^2. \tag{16}$$

This is reasonable under the assumption that the strong forward peak of the phase function has a small width $\langle\theta^2\rangle^{1/2}$, and that after several scattering lengths the radiance distribution is no longer sensitive to the detailed character of the phase function. We then expand the path integral over $\vec{p}(s')$ around the function for which the integrand is maximum. This function is

$$\vec{p}_0(s') = i\frac{(1-\exp(-bs))}{\langle\theta^2\rangle\, b}\frac{d\hat{\beta}(s')}{ds'}\ . \tag{17}$$

In this expansion, the path integral over $\vec{p}$ has a gaussian form and can be evaluated.

The remaining path integral over $\hat{\beta}$ must be evaluated carefully to insure that $\hat{\beta}$ remains a unit vector under any approximations applied, so that photons arrive at the correct times. Since the problem has been restricted to computing moments along the beam axis, $\hat{\beta}$ maintains the proper form under the representation

$$\hat{\beta}(s') = \Re(\theta(s'),\phi(s'),\psi(s'))\cdot\hat{n}_0\ , \tag{18}$$

where $\Re$ is a rotation matrix and θ,ϕ, and ψ are the Euler angles of an arbitraty rotation. The boundary conditions on $\hat{\beta}$ are satisfied by the expansion of the Euler angles in Fourier series:

$$\begin{bmatrix}\theta(s')\\ \phi(s')\\ \psi(s')\end{bmatrix} = \sum_{n=1}^{\infty}\begin{bmatrix}a_\theta^n\\ a_\phi^n\\ a_\psi^n\end{bmatrix}\sin(\pi n s'/s)\ . \tag{19}$$

In fact, given the constraint equation

$$\vec{x} = \int_0^s ds' \, \hat{\beta}(s') \; , \tag{20}$$

the simplest solution is

$$\begin{aligned} a_\theta^n &= 0 & (21)\\ a_\phi^n &= 0 & (22)\\ a_\psi^n &= 0 \quad (n \neq 2) & (23) \end{aligned}$$

and the full expression reduces to

$$\begin{aligned} L(s, x\,\hat{n}_0, \hat{n}_0) &= \int_0^\infty da_\psi \, \delta\left(x - sJ_0(a_\psi)\right) \frac{(1-\exp(-bs))}{2\pi \langle \theta^2 \rangle bs^3} \exp(-cs) \\ &\times \exp\left\{ -\frac{(1-\exp(-bs))}{\langle \theta^2 \rangle bs} \frac{(2\pi a_\psi)^2}{2} \right\} \\ &\times \left(\exp\left\{ bs + \frac{(1-\exp(-bs))^2}{2\langle \theta^2 \rangle bs} \frac{(2\pi a_\psi)^2}{2} \right\} - 1 \right) \end{aligned} \tag{24}$$

The delta function in this expression is the remnants of the constraint condition in equation 20. Note that since the Bessel function $J_0 \leq 1$, the constraint guarantees the photons do not arrive too early.

2.3 Moments

From equation 24, the generating function can easily be obtained because the delta function allows only the time $s = s_0$ for which

$$\frac{x}{s_0} = J_0(a_\psi) \tag{25}$$

to contribute. Substituting this expression for s into equation 24, and expanding the Bessel function as

$$J_0(a_\psi) \approx 1 - \frac{1}{4} a_\psi^2 \; , \tag{26}$$

the generating function is (ignoring constant factors)

$$\mathcal{Z}(a, x) = \frac{1}{x^3} \exp(-cs) \sum_{i=1}^{4} \frac{\alpha_i}{F_i^{1/2}} \; . \tag{27}$$

The quantities α_i and F_i are

$$\begin{aligned} \alpha_1 &= \exp(bx) \\ \alpha_2 &= -1 \\ \alpha_3 &= -1 \\ \alpha_4 &= \exp(-bx) \end{aligned}$$

and

$$
\begin{aligned}
F_1 &= \frac{ax}{2} + \frac{(1-\exp(-2bx))}{2\langle\theta^2\rangle bx}(2\pi)^2 \\
F_2 &= \frac{cx}{2} + \frac{(1-\exp(-bx))}{\langle\theta^2\rangle bx}(2\pi)^2 \\
F_3 &= \frac{cx}{2} + \frac{(1-\exp(-2bx))}{2\langle\theta^2\rangle bx}(2\pi)^2 \\
F_4 &= \frac{cx}{2} + \frac{bx}{2} + \frac{(1-\exp(-bx))}{2\langle\theta^2\rangle bx}(2\pi)^2
\end{aligned}
$$

3 Pulse delay and width

From the generating function in equation 27, the pulse delay time is

$$\Delta s(x) = \frac{x}{4}\frac{\sum_{i=1}^4 \alpha_i F_i^{-3/2}}{\sum_{i=1}^4 \alpha_i F_i^{-1/2}} \tag{28}$$

and the width is given by

$$\sigma_s^2(x) = \left(\frac{x}{4}\right)^2 \left\{ 3\frac{\sum_{i=1}^4 \alpha_i F_i^{-5/2}}{\sum_{i=1}^4 \alpha_i F_i^{-1/2}} - \left(\frac{\sum_{i=1}^4 \alpha_i F_i^{-3/2}}{\sum_{i=1}^4 \alpha_i F_i^{-1/2}}\right)^2 \right\} \tag{29}$$

A more accurate alternative to using equation 27 is to evaluate the integral over a_ψ numerically, solving numerically for s_0 as well. Both methods have been used, and the analytic results in equations 28 and 29 are very accurate. For example, figure 1 shows $c\ \Delta s$ as a function of the extinction lengths cx for single scatter albedo $\omega_0 = 0.6$. Both numerical and analytical evaluations are plotted. Similarly, figure 2 shows the width in units of extinction lengths as a function of the number of extinction lengths. Again, both numerical and analytical evaluations are plotted.

Note that the shape of the delay and width are similar in figures 1 and 2. In the multiple scattering limit $bx \gg 1$, the delay and width are proportional

$$\sigma_s \approx 2^{1/2}\Delta s\ . \tag{30}$$

In fact this relationship is a generally reasonable one. Figure 3 shows the error e of this assumption, defined as

$$e = \frac{\sigma_s^2}{2(\Delta s)^2} - 1\ . \tag{31}$$

The error is never larger in magnitude than 12%.

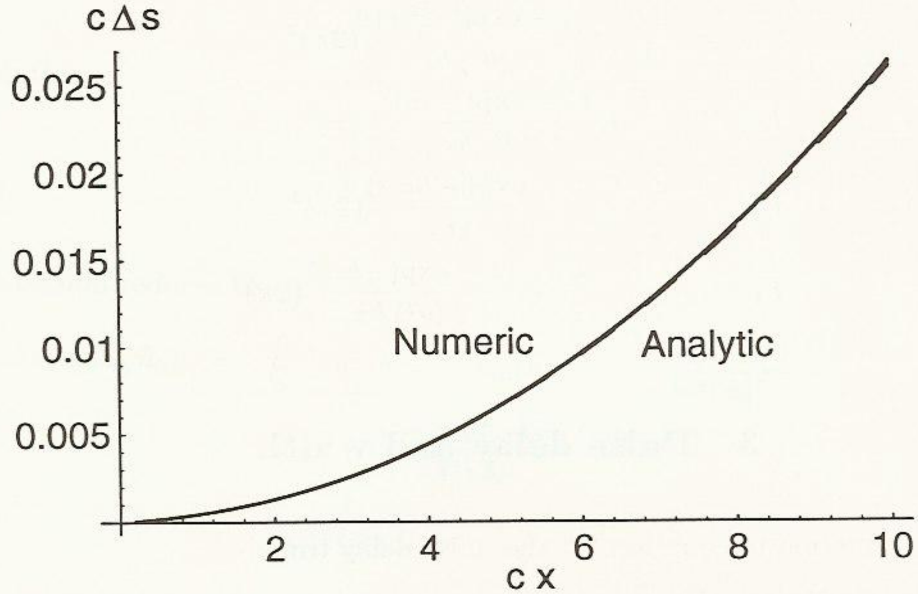

Figure 1: Plots of the temporal delay $c\,\Delta s$ of a pulse as a function of the number of extinction lengths cx. The single scatter albedo is $\omega_0 = 0.6$ and the phase function width is $\langle\theta^2\rangle = 0.035$. Both numerical (solid) and analytical (dashed) evaluations are plotted.

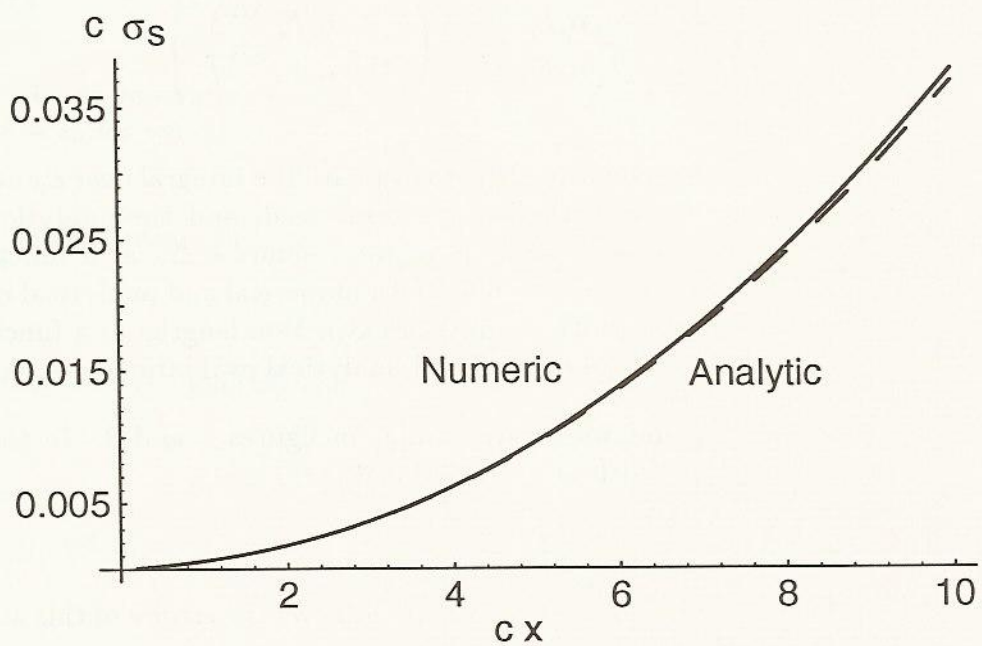

Figure 2: Plots of the temporal width $c\,\sigma_s$ of a pulse as a function of the number of extinction lengths cx. The single scatter albedo is $\omega_0 = 0.6$ and the phase function width is $\langle\theta^2\rangle = 0.035$. Both numerical (solid) and analytical (dashed) evaluations are plotted.

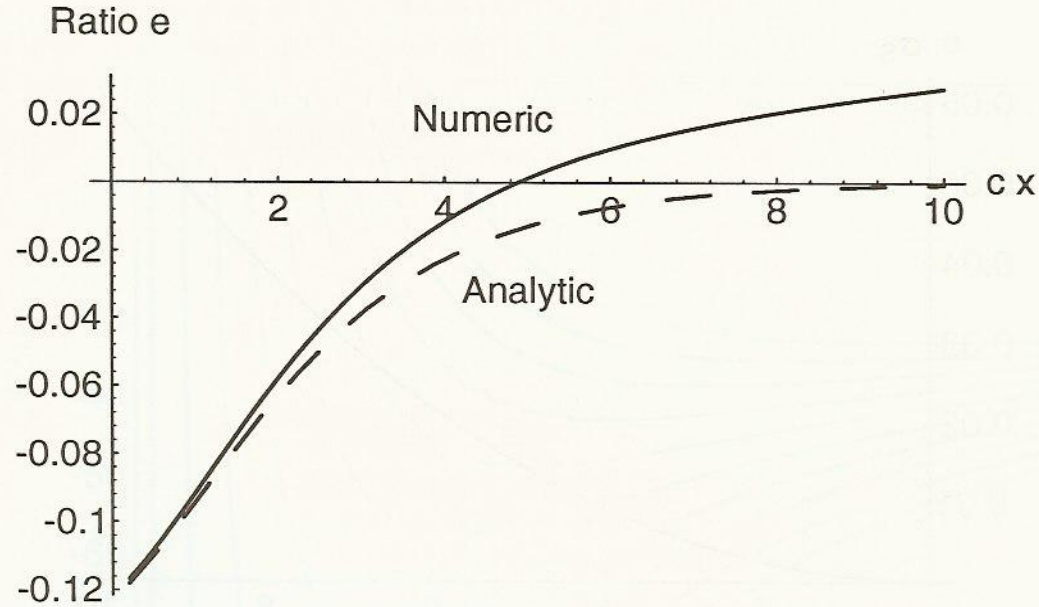

Figure 3: Plot of the ratio $e = \sigma_s^2/2(\Delta s)^2 - 1$ as a function of the number of extinction lengths cx. The single scatter albedo is $\omega_0 = 0.6$ and the phase function width is $\langle\theta^2\rangle = 0.035$. Both numerical (solid) and analytical (dashed) evaluations are plotted.

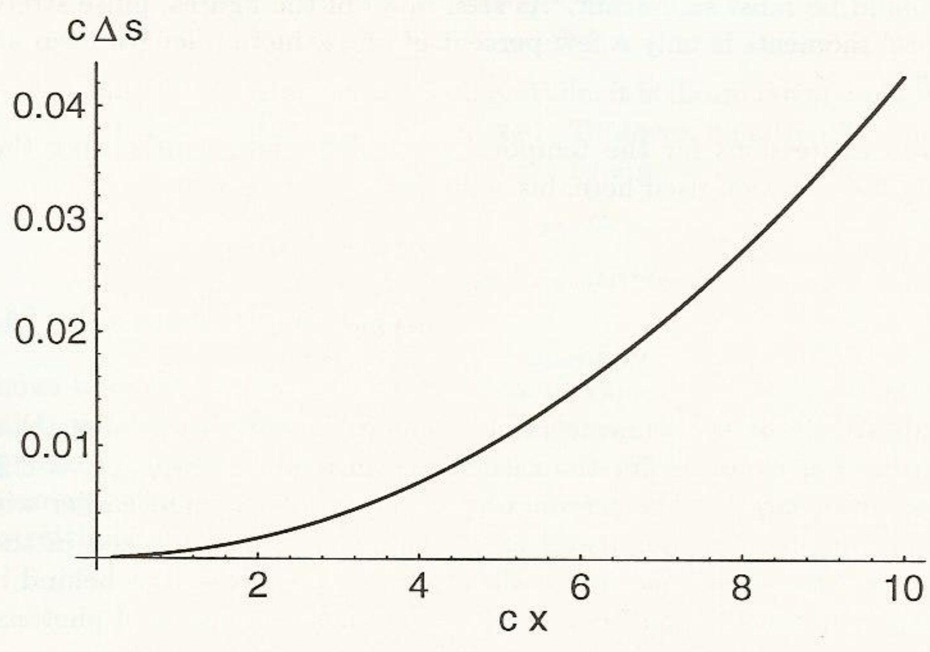

Figure 4: Plot of the numerically calculated delay $c\ \Delta s$ as a function of the number of extinction lengths cx. The single scatter albedo is $\omega_0 = 0.95$ and the phase function width is $\langle\theta^2\rangle = 0.035$.

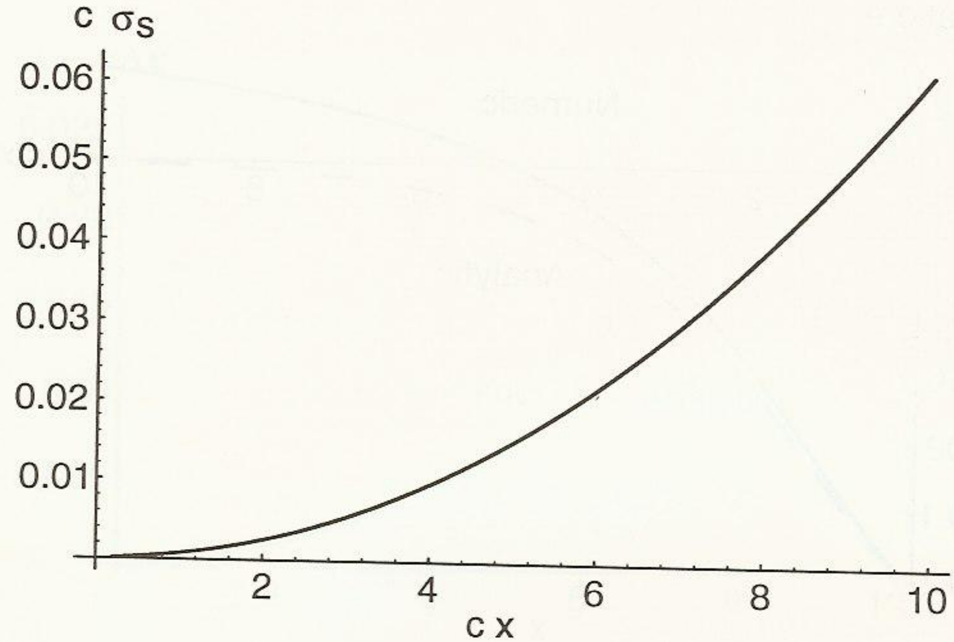

Figure 5: Plot of the numerically calculated width $c\sigma_s$ as a function of the number of extinction lengths cx. The single scatter albedo is $\omega_0 = 0.95$ and the phase function width is $\langle\theta^2\rangle = 0.035$.

Figures 4 and 5 show the delay and width for single scatter albedo $\omega_0 = 0.95$, a regime in which pulse stretching should be most significant. As seen in all of the figures, pulse stretching defined in terms of temporal moments is only a few percent of an extinction length even at distances of 10 extinction lengths.

Ishimaru derived expressions for the temporal pulse delay and width using the small-angle approximation[4]. In the notation used here, his solution is

$$\Delta s_{Ishimaru} = \frac{\langle\theta^2\rangle\, bx^2}{2} \quad (32)$$

$$\sigma_{Ishimaru} = \frac{\langle\theta^2\rangle\, bx^2}{4}\,. \quad (33)$$

This result is qualitatively of the same form, but quantitatively much larger than the values plotted in the figures. For example, for the case shown in figure 2 $c\sigma_{Ishimaru} = 0.5$ at $cx = 10$, more than a factor of 10 larger. The reason why is the problem noted earlier with the small-angle approximation: photons do not travel at the physically correct speed in the small-angle approximation, so that stretching takes place ahead of the pulse as well as behind it. The width of the pulse then appears much larger because of this unphysical leakage of photons through the light cone.

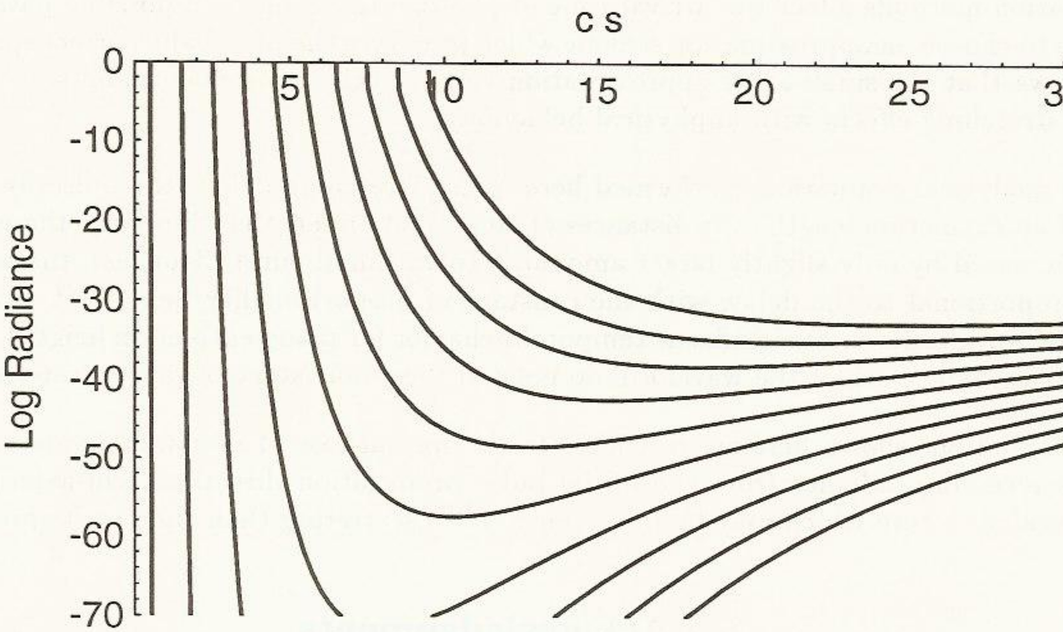

Figure 6: Log plot of waveforms for radiometers spaced one extinction length apart as a function of time cs measured in units of extinction length. The single scatter albedo is $\omega_0 = 0.6$ and the phase function width is $\langle \theta^2 \rangle = 0.035$.

4 Waveforms

Although the delay and width are not large in magnitude, it is illuminating to look at the temporal waveforms from which these statistics are calculated. To do so, equation 24 can be evaluated by solving for the smallest root a^*_ψ of the delta function. To a good approximation, this solution is

$$a^*_\psi = a^0_\psi \left(1 - \frac{x}{s}\right)^{1/2}, \tag{34}$$

where $a^0_\psi \approx 2.47$ is the smallest root of J_0.

Figure 6 shows temporal waveforms from radiometers spaced one extinction length apart along the beam axis. It is clear from this plot that the scattered component of the pulse evolves to a significant degree over distances of 10 extinction lengths. We should expect therefore that the higher order moments that have not been calculated here play an important role in characterizing the temporal stretching of a pulse.

5 Conclusions

The path integral formalism provides a powerful analytical starting point for investigating the temporal evolution of pulses in ocean water. One useful feature is that it explicitly shows how approximation methods affect the arrival time of photons traveling each possible path. This has allowed us to choose an approximation scheme which preserves the physically correct speed of light. It also shows that the small-angle approximation violates this requirement, and so contaminates the pulse stretching effects with unphysical behavior.

In the analytical evaluations performed here, pulse stretching delays the pulse by only a few percent of an extinction length over distances of 10 extinction lengths. Similarly, the width of the pulse is increased by only slightly larger amounts. To within about 12% or less, the width of the pulse is proportional to the delay, with the constant of proportionality being $2^{1/2}$. However, the structure of waveforms show significant temporal behavior for many extinction lengths, suggesting that the delay and width of the waveform do not tell the whole story of pulse stretching.

The calculations shown here are restricted to the special case of radiometers placed along the pulse axis receiving radiance from the initial pulse propagation direction. Off-axis radiometers should reveal structure which may be more sensitive to scattering than the results presented.

6 Acknowledgements

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