Approximate parametric receiver operating characteristics for Poisson distributed noise

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Received 26 May 1988.
0003-6935/89/020214-03$02.00/0.

Radiometric receivers based on photon-counting techniques are subject to photon-counting noise as the basic limit to their signal detection performance. In some circumstances in which the number of photons received in a given time interval is large, the Poisson distribution of photon counting is effectively Gaussian and the receiver operating characteristics (ROC) may be based on standard error function expressions.[1] In more stressing conditions the non-Gaussian tail of the Poisson distribution becomes significant and the ROC must be obtained by an alternative method. If there is only one channel, the probabilities of detection and false alarm are

$$P_d(n_T) = 1 - \sum_{n=0}^{n_T-1} \frac{(S+B)^n}{n!} \exp\{-(S+B)\},$$

$$P_{fa}(n_T) = 1 - \sum_{n=0}^{n_T-1} \frac{B^n}{n!} \exp\{-B\},$$

where B is the mean count level expected when a signal is not present, S is the mean count level in excess of B when a signal is present, and n_T is the chosen threshold count level for a detection declaration.

When the threshold is not too large these expressions can be evaluated numerically even on a pocket calculator.[2,3] However, when the threshold is large, or when there are several channels combined into a test statistic, direct evaluation of the sums becomes difficult and time-consuming. Gagliardi and Karp developed ROC expressions for the detection of a signal which may occur in any one of M channels.[4] In that case, however, the decision was a choice between M hypotheses, giving operating characteristics different from those of the classic binary decision problem. In addition, the expressions they obtained for P_d and P_{fa} involved infinite series, so that the difficulty in numerical evaluation was not overcome.

In this Letter approximate expressions are given for P_d and P_{fa} as functions of a threshold, and a derivation of the expressions is outlined. Asymptotic conditions for the validity of the approximation are given, and a comparison with an exact evaluation shows that these conditions are stronger than need be at high threshold. The expressions are simple enough to be evaluated on a pocket calculator.

The basic approach used here was developed by Thebaud,[5] who constructed approximate expressions using analytic continuation and a stationary phase approximation on the integrals displayed below. The approach taken here is a steepest-descents approximation. The results are an extension of those in Ref. 5 and are more well behaved at small threshold levels.

Suppose a receiver system outputs N channels with counts n_i, $i = 1, \ldots, N$ (e.g., the channels may be multiple radiometers or successive samples in a time series). The photon counting noise in each channel independently satisfies a Poisson distribution with probability density

$$\mathcal{P}_i(n_i) = \frac{\langle n_i\rangle^{n_i}}{n_i!} \exp(-\langle n_i\rangle),$$

where $\langle n_i\rangle$ is the mean expected count level in channel i. The probability density for the combination of the N channels is the product of the individual densities. The test statistic used for declaring the detection of a signal is the log-likelihood ratio from classical binary detection theory

$$\lambda(n_i, \ldots, n_N) = \sum_{i=1}^{N} (n_i \xi_i - S_i),$$

with

$$\xi_i = \log\left(1 + \frac{S_i}{B_i}\right).$$

B_i and S_i are as defined previously but may have different values for each channel. The detection and false alarm probabilities for a given threshold λ_T are (θ is the Heaviside step function)

$$P_d(\lambda_T) = \sum_{\{n_i\}=0}^{\infty} \prod_{i=1}^{N} \frac{(S_i + B_i)^{n_i}}{n_i!} \exp\{-(S_i + B_i)\}\theta[\lambda(n_1, \ldots, n_N) - \lambda_T],$$

$$P_{fa}(\lambda_T) = \sum_{\{n_i\}=0}^{\infty} \prod_{i=1}^{N} \frac{B_i^{n_i}}{n_i!} \exp\{-B_i\}\theta[\lambda(n_1, \ldots, n_N) - \lambda_T].$$

By using the Fourier transform identity for the step function ($\epsilon \rightarrow 0^+$),

$$\theta(x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dq}{q + i\epsilon} \exp(-iqx),$$

the summation over the ensemble of counts can be evaluated exactly to give

$$P_r(\lambda_T) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dq}{q + i\epsilon} \exp\{iq\lambda_T + \psi_r(q)\},$$

where $r = d, fa$, and

$$\psi_d(q) = \sum_{j=1}^{N} \{iqS_j + (S_j + B_j)[\exp(-iq\xi_j) - 1]\},$$

$$\psi_{fa}(q) = \sum_{j=1}^{N} \{iqS_j + B_j[\exp(-iq\xi_j) - 1]\}.$$

These are the integral expressions obtained by Thebaud in Ref. 5 and approximately evaluated by the analytic continuation $q \rightarrow iq$ and a stationary phase evaluation. The results of the approximation are valid for thresholds that are very large compared to the average of the test statistic $\lambda(n_1, \ldots, n_N)$.

The approximate evaluation below proceeds via the steepest-descents method, in which the integration contour is deformed from the real axis to the complex plane. The deformed contour does not cross the pole at the origin and fixes the imaginary component of

$$F_r(q) = i\lambda_{T_q} + \psi_r(q) - \log(q) + i\pi/2$$

to be constant. Since the integral is real, the simplest path to choose has the phase $Im(F_r) = 0$. Along the constant phase path the integral may be approximated by Laplace's method.[6] The extremum point is on the imaginary axis away from the origin, and the steepest-descents path in the neighborhood of the extremum point is a parabola oriented upward. The result for the probability of detection is

$$P_d(\lambda_T) = \left\{2\pi\left[1 + Q^2 \sum_{j=1}^{N} \xi_j^2(S_j + B_j) \exp(Q\xi_j)\right]\right\}^{-1/2} \exp(F_d),$$

with

$$F_d = -Q\left(\lambda_T + \sum_{j=1}^{N} S_j\right) + \sum_{j=1}^{N} (S_j + B_j)[\exp(Q\xi_j) - 1],$$

and Q is obtained implicitly from the extremum condition

$$\lambda_T = \sum_{j=1}^{N} \{\xi_j(S_j + B_j) \exp(Q\xi_j) - S_j\} - \frac{1}{Q}\cdot$$

Similarly, for the probability of false alarm,

$$P_{fa}(\lambda_T) = \left\{2\pi\left[1 + Q^2 \sum_{j=1}^{N} \xi_j^2 B_j \exp(Q\xi_j)\right]\right\}^{-1/2} \exp(F_{fa}),$$

with

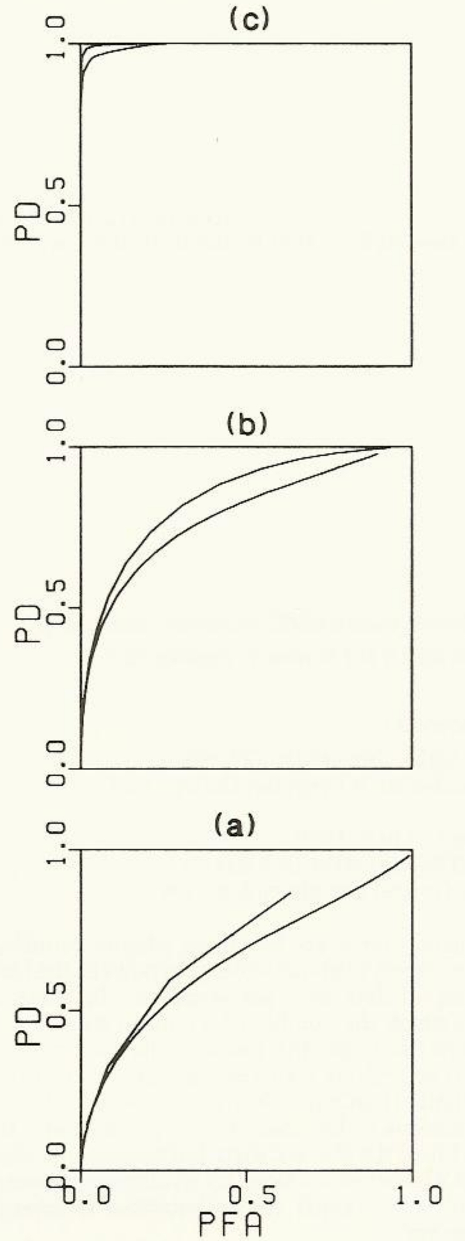

Fig. 1. Comparisons between approximate and exact evaluations of the ROC for one channel. The upper curve in each comparison is the exact ROC, and the lower curve is the approximate ROC. (a) $S = 1, B = 1$; (b) $S = 5, B = 10$; (c) $S = 5, B = 40$.

$$F_{fa} = -Q\left(\lambda_T + \sum_{j=1}^{N} S_j\right) + \sum_{j=1}^{N} B_j[\exp(Q\xi_j) - 1],$$

and now the equation for Q is

$$\lambda_T = \sum_{j=1}^{N} \{\xi_j B_j \exp(Q\xi_j) - S_j\} - \frac{1}{Q}\cdot$$

Note that the range $0 < Q < \infty$ corresponds to the range $-\infty < \lambda_T < \infty$, so that in principle the entire ROC curve can be obtained. In practice, however, the approximation breaks down for small Q, as can be seen by the fact that the probabilities at $\lambda_T = -\infty$ are >1:

$$P_d(-\infty) = P_{fa}(-\infty) = \left(\frac{e^2}{2\pi}\right)^{1/2} \approx 1.084437551.$$

If the mean number of counts is large, the value of Q at which $P_d = 1$ is approximately

$$Q^2 = \frac{1}{3}\log\left(\frac{e^2}{2\pi}\right)\left\{\sum_{j=1}^{N} \xi_j^2(S_j + B_j)\right\}^{-1}.$$

The range of validity of the approximation is obtained by requiring higher terms of the steepest-descent approximation to be small. The conditions imposed are

$$Q^4 \gg 1/\max[\xi_j^4(S_j + B_j)],$$

$$Q^2 \gg 1/\max[\xi_j^2(S_j + B_j)],$$

$$\max(S_j) > 0,$$

$$B_j \gg 1 \text{ for all } j.$$

The second condition guarantees that the range of Q for which the probabilities are greater than unity is outside the range of validity of the approximation.

The examples in Fig. 1 show that the last condition that the mean count level be large is not a strong condition. In particular, the approximate expression is within 15% of the exact result even for $S = 1, B = 1$. Best results are achieved when the threshold is large compared to the averaged test statistic $\langle\lambda(n_1, \ldots, n_N)\rangle$.

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