Multiple-Forward-Scattering in Volume Rendering of Participating Media

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Abstract

Natural volumetric media have phase functions which typically are sharply peaked in the forward scattering direction, with backscatter accounting for only a few percent of the total angular redistribution from a single scattering event. This property has been exploited in the past in the small-angle approximation for radiative transfer, successfully for many engineering and science applications. The small-angle approximation also robustly describes the multiple-forward-scattered behavior of the light field many scattering lengths into the participating medium, including the asymptotic regime, in agreement with experimental measurements and computationally intensive simulations. Physically, the important missing ingredient not found in the small-angle approximation is occasional large angle scatters that reverse the propagation direction of some of the light. This paper introduces a quantitative model of multiple scattering which contains both the multiple-forward-scatter character and a few large-angle scattering events. The model is derived directly from the Green's function representation of radiative transfer, and path integrals are used to construct the appropriate form of the small angle approximation. The model is suitable for media that have internal structure. This paper reports on a work in progress, presenting the model but no rendered results.

1 Introduction

Many naturally occurring participating media share a common trait: their scattering phase functions are sharply peaked in the forward direction in visible light. In all cases, the value of the phase function near 0 degrees is 10-1000 times higher than the peak in the backscatter direction. The source of this forward-peaked character is the particulate nature of the media; the particles are generally much larger than the wavelength of visible light, which is a regime that favors forward scattering. An important exception to this is the molecular medium of the atmosphere, in which oxygen- and nitrogen-based molecules are comparable in size to visible light. In this limit, Rayleigh scattering dominates and the phase function is very broad. This paper is concerned more with particulate media that have forward peaked phase functions.

This quality of the phase function is important for multiple scattering. For example, a volume that is 100 scattering lengths thick (e.g. a cumulus cloud or a human hand), only a few of those scattering events are large angle scatters, while the great majority of them are scatters through a small angle from the incoming direction. The diffusive look of a cloud or tissue arises mostly from the multiple forward scatters, with the few large angle scatters playing a "secondary" role.

The small-angle approximation [Dol80] to the radiative transfer equation focuses on just the multiple-forward-scatter behavior. For a light beam incident on a participating medium, the radiance distribution at a distance z into the medium is proportional to

$$\exp\left(-\frac{\vec{n}^2}{2\langle\theta^2\rangle bz}\right)$$

where $\vec{n}$ is the component of the direction vector perpendicular to the incident direction, $\langle\theta^2\rangle$ is the square of the width of the forward peak of the phase function, and b is the scattering coefficient. This form is a diffusion of the light in the angular degrees of freedom at depths in the medim. In addition, the radiance distribution is proportional to

$$\exp\left(-\frac{\vec{r}^2}{2\langle\theta^2\rangle bz^3}\right)$$

and $\vec{r}$ is the component of position in the medium perpendicular to the incident direction. This is a diffusion in the spatial structure of the the light field.

This form of the small angle approximation is useful for some applications because of its simplicity. A re-examination of the approximation procedure by [Tes87] produced a version that is slightly more complex, but much more accurate, especially at great depths. In this improved small angle approximation, the angular and spatial diffusion are replaced with the altered factors:

$$\exp\left(-\vec{n}^2\frac{\sinh(2q)}{4\langle\theta^2\rangle b\ell\sinh^2(q)}\right)$$

and

$$\exp\left(-\frac{(\vec{r}-\vec{n}\ell R(q))^2}{2\langle\theta^2\rangle b\ell^3 h(q)}\right)$$

In this form, a new parameter ℓ, which has dimensions of lenght, comes into the picture. Its definition is $1/\ell^2 = \langle\theta^2\rangle ab$, combining the scattering coefficient b, the absorption coefficient a, and the width of the phase function peak $\langle\theta^2\rangle$. The dimensionless quantity q is z/ℓ, and the functions h and R reduce to the previous small angle approximation in the limit $z \ll \ell$ [Tes88].

The length scale ℓ arises from this approximation as the marker between a purely diffusive regime over short distances ($z < \ell$) and the so-called asymptotic regime ($z > \ell$) studied by [HZ77], for which the light distribution is effectively frozen in shape and decays exponentially with depth according to a diffuse attenutation coefficient. This improved small angle approximation compares well with experimental and simulation data [Tes88] in an angular region around the initial propagation direction, and shows very well the transition to the asymptotic regime. Note that the original form of the small angle approximation does not predict the existence of an asymptotic regime at all. This more accurate second form of the small angle approximation will be used in the remainder of this paper.

The small angle approximation successfully captures the multiple forward scattering behavior that comes from phase functions that are sharply peaked in the forward direction. The remainder of the light distribution coming from a participating medium comes from large angle scattering events. Only a few of these scattering events need occur, even within optically thick media, for a much more accurate rendering of the full light field. The aim of this paper is to fuse the small angle approximation for multiple forward scattering with a discrete scatter model. The fusion qualitatively consists of having light propagating through the medium in a multiple-forward-scatter process, suffering a large angle scatter at a point in the medium volume, then continuing to propagate in the new direction with multiple-forward-scattering. By iterating this basic

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picture, more and more large angle scatters can be incorporated as long as it is computationally practical to do so.

To accomplish this fusion, there are two tasks. The first one is to build an integral form of the radiative transfer equation that allows us to select large angle scattering events separate from multiple forward scatters (section 3). The second task is to rebuild the small angle approximation in a form that is more suitable for general volume rendering applications. The form discussed above is rigidly tied to half-space geometries. We recast it in section 4 to be more suitable for general volume rendering problems. The final fused algorithm has the form of a raymarching process. As the ray march proceeds deeper into the medium, the spatial and angular blurring from the diffusion can be exploited to increase the step size of the raymarch.

2 Background

There are several approaches to multiple scattering in use in computer graphics. The subsurface scattering approach of [DJ05] is design specifically for geometries and material properties that support the diffusion approximation for the radiance leaving the surface of the medium. Multiple scattering has been implemented in hardware volume rendering by [HL01], who used to concept of multiple forward scattering to motivate using textured cards in the field of view, and [REK*04], who implemented multiple scattering in hardware by building a look up table of phase function convolutions to describe multiple scatters.

Multiple forward scattering has also been exploited by [PAS03] and in hardware by [HPAD06], with algorithms based on the small angle scattering approximation. This approach was motivated by the path integral methods for solving the radiative transfer equation first introduced by [Tes87] for underwater optics, and later by [PaIF94] for tissue optics.

3 Green's Function

We have chosen to build a fused algorithm using the Green's function for the radiative transfer equation. Imagine we begin with a dark medium and turn on a focused laser light source emitting a very short pulse from any point $\vec{x}'$ in the medium in any direction $\hat{n}'$. Scattering and attenuation redistribute the light in the volume, and at a time s after emission, the light arriving at any other point $\vec{x}$ in a direction $\hat{n}$ is proportional to the Green's function, written $G(s,\vec{x},\hat{n},\vec{x}',\hat{n}')$. In principle, this function is the solution of the time-dependent radiative transfer equation

$$\left\{\frac{\partial}{\partial s} + \hat{n}\cdot\nabla + c(\vec{x})\right\} G(s,\vec{x},\hat{n},\vec{x}',\hat{n}') \quad = \quad b(\vec{x}) \int d\Omega''\, P(\hat{n},\hat{n}'')\, G(s,\vec{x},\hat{n}'',\vec{x}',\hat{n}')$$

with the initial condition

$$G(0,\vec{x},\hat{n},\vec{x}',\hat{n}') \;=\; \delta\left(\vec{x}-\vec{x}'\right)\,\delta\left(\hat{n}-\hat{n}'\right)$$

For light sources that stay on indefinitely, the radiance distribution L is the convolution of the Green's function with the light source:

$$L(\vec{x},\hat{n}) = \int_0^\infty ds \int d^3x'\, d\Omega'\, G(s,\vec{x},\hat{n},\vec{x}',\hat{n}')\, S(\vec{x}',\hat{n};)$$

This solution for the radiative transfer problem is naturally implemented as a ray march algorithm. The integral over time serves as the ray march stepping mechanism. This will be made explicit below.

Using the time-dependent Green's function to find a time-independent light field lets us exploit certain technical simplifications, because the time parameter effectively sorts all of the light arriving at a point into packets that arrive at different times. This also helps to build approximation strategies for multiple forward scattering.

The Green's function handles both scattered and unscattered light. To get at strategies for scattering, we first split it into a term for unscattered light and a term for scattered light:

$$\begin{aligned} G(s,\vec{x},\hat{n},\vec{x}',\hat{n}') &= \delta\left(\vec{x}-\vec{x}'-\hat{n}s\right)\,\delta\left(\hat{n}-\hat{n}'\right)\, T(0,s,\vec{x},\hat{n}) \\ &+ \Delta G(s,\vec{x},\hat{n},\vec{x}',\hat{n}') \end{aligned}$$

and T is the exponential attenuation due to the extinction coefficient between $\vec{x}$ and $\vec{x}'$:

$$T(s',s,\vec{x},\hat{n}) = \exp\left(-\int_{s'}^{s} ds''\, c\left(\vec{x}-\hat{n}(s-s'')\right)\right)$$

Because the unscattered light term satisfies the initial conditions for the Green's function, the initial conditions for the scattered light Green's function is that there is no scattered light initially, i.e. $\Delta G(s=0)=0$.

The scattered Green's function satisfies an integral equation consiting of two terms, one for single scattering and one for multiple scattering:

$$\Delta G = \Delta G_{SS} + \Delta G_{MS}$$

where the first term is a single scatter expression

$$\begin{aligned} \Delta G_{SS}(s,\vec{x},\hat{n},\vec{x}',\hat{n}') &= \int_0^s ds'\, T(0,s-s',\vec{x},\hat{n}) \\ &\times\; b\left(\vec{x}-\hat{n}(s-s')\right)\, P(\hat{n},\hat{n}') \\ &\times\; T(0,s',\vec{x}-\hat{n}(s-s'),\hat{n}') \end{aligned} \tag{1}$$

and the second term is for more than one scatter:

$$\begin{aligned} \Delta G_{MS}(s,\vec{x},\hat{n},\vec{x}',\hat{n}') &= \int_0^s ds'\, d\Omega'' \\ &\times\; \Delta G(s-s',\vec{x},\hat{n},\vec{x}'+\hat{n}'s',\hat{n}'') \\ &\times\; b\left(\vec{x}'+\hat{n}'s'\right)\, P(\hat{n}'',\hat{n}') \\ &\times\; T(0,s',\vec{x}'+\hat{n}'s',\hat{n}') \end{aligned} \tag{2}$$

These two contributions to ΔG tell relatively simple stories. The first one, ΔG_{SS}, is the single scattering of light from the direction of the light source to the direction of the viewer. From the light source to the scattering point, the light suffers attenuation $T(0,s',\vec{x}-\hat{n}(s-s'),\hat{n}')$. In some implementations of volume rendering, this factor is included in a deep shadow map volumetric data structure. The phase function scatters the light from the direction $\hat{n}'$ to $\hat{n}$, and from the scatter point to the viewer there is attenuation $T(0,s-s',\vec{x},\hat{n})$. Because the single scattering event may occur at many possible points along rays from the light source, the integration $\int ds'$ finds the appropriate intersection point for each ray. This single scattering term is commonly evaluated as a raymarch from the viewer into the volume, with the integration variable s' playing the role of stepping the ray through the volume.

Multiple scattering occurs in ΔG_{MS}. This term has similarities to the single scattered one. Propagation from the light source is accompanied by an attenuation $T(0,s',\vec{x}'+\hat{n}'s',\hat{n}')$, which again may be incorporated into a deep shadow map. There is a single scatter of the light at the point $\vec{x}'+\hat{n}'s'$, and the light which leaves there and arrives at the viewer is then scattered more by the scattered Green's function ΔG. This term is the primary focus of this paper. Notice that it has the form that is desired from the discussion in section 1: there is scatter, which may be a large angle scatter, followed by other scatters. Since the first scatter fulfills the need of incorporating a large angle scatter, the ΔG factor in the convolution can reasonably be approximated by a multiply-forward-scattered expression. The approach in this paper is to build an expression for the multiple forward scattered Green's function, ΔG_{MFS} based on the small angle approximation discussion above, and the multiple scattered Green's function is

$$\begin{aligned}
\Delta G_{MS}(s,\vec{x},\hat{n},\vec{x}',\hat{n}') &= \int_0^s ds'\, d\Omega'' \\
&\times \Delta G_{MFS}(s-s',\vec{x},\hat{n},\vec{x}'+\hat{n}'s',\hat{n}'') \\
&\times b\left(\vec{x}'+\hat{n}'s'\right)\, P(\hat{n}'',\hat{n}') \\
&\times T(0,s',\vec{x}'+\hat{n}'s',\hat{n}') \qquad (3)
\end{aligned}$$

4 Multiple Forward Scattering Green's Function

The small angle approximation that we use here is based on the one introduced by [Tes87] using path integral methods. That form, however, is not sufficient for us for three reasons: (1) the original form does not separate scattered and unscattered light for applying the small angle approximation, so the solution in [Tes87] will not satisfy our initial conditions; (2) when the unscattered light is removed prior to the small angle approximation, some of the quantities are altred in important ways; (3) the derivation used in [Tes87] assumed a plane-parallel geometry in which the medium occupied a semi-infinite volume, while we need a more general structure for the medium.

In this section we run through the path integral formulation and small angle approximation to arrive at an expression for the multiple forward scattered Green's function ΔG_{MFS}. The result is valid for a general participating medium. For simplicity in the explanation, a homogeneous medium is used, then in section 5 we discuss the changes to the result that are needed to make in valid in arbitrary media.

The path integral expression for the scattered Green's function is an exact but formal solution of the radiative transfer problem in terms an an infinite-dimensional integral over the phase space all all possible paths through the medium and all possible scattering modes. Following [Tes89], it is

$$\begin{aligned}
\Delta G(s,\vec{x},\hat{n},\vec{x}',\hat{n}') &= \int d\mu(\hat{\beta},\vec{p})\, \delta\left(\hat{\beta}(0)-\hat{n}'\right)\, \delta\left(\hat{\beta}(s)-\hat{n}\right) \\
&\times \delta\left(\vec{x}-\vec{x}'-\int_0^s ds'\, \hat{\beta}(s')\right) \exp(-cs) \\
&\times \exp\left(i\int_0^s ds'\, \vec{p}(s')\cdot\frac{d\hat{\beta}(s')}{ds'}\right) \\
&\times \left(\exp\left(i\int_0^s ds'\, b\, Z(\vec{p}(s'))\right)-1\right) \qquad (4)
\end{aligned}$$

The unit vector $\hat{\beta}(s')$ describes all possible paths through the medium that start at $(\vec{x}',\hat{n}')$ and end at $(\vec{x},\hat{n})$ after a time s. Specifically, it is the tangent to the paths.

Multiple forward scattering is enforced by approximating the Fourier-transformed phase function, Z, using the fact that the phase function has a very strong and narrow peak around the forward direction. In this situation, the dominant contribution of scattering to the paths comes from setting

$$Z(\vec{p}) \approx 1 - \frac{\langle\theta^2\rangle}{2}\,|\vec{p}|^2 \qquad (5)$$

with the understanding that $\langle\theta^2\rangle$ has a small value. In this situation, the path integral over the scattering modes $\vec{p}$ can be evaluated, leaving only the path integral over the paths $\hat{\beta}$. This integral has the form

$$\begin{aligned}
\Delta G_{MFS} &= \int d\mu(\hat{\beta})\, \delta\left(\hat{\beta}(0)-\hat{n}'\right)\, \delta\left(\hat{\beta}(s)-\hat{n}\right) \\
&\times \delta\left(\vec{x}-\vec{x}'-\int_0^s ds'\, \hat{\beta}(s')\right) \exp(-as) \\
&\times \left(1-e^{-bs}\right) \\
&\times \exp\left(-\frac{\left(1-e^{-bs}\right)}{2\langle\theta^2\rangle b}\int_0^s ds' \left|\frac{d\hat{\beta}(s')}{ds'}\right|^2\right) \qquad (6)
\end{aligned}$$

Since light travels at a fixed speed, the scattered Green's function should enforce that speed even in approximate evaluations. In fact, it is easy to see the fixed speed constraint directly from the full exact path integral expression. The spatial delta function

$$\delta\left(\vec{x}-\vec{x}'-\int_0^s \hat{\beta}(s')ds'\right)$$

requires that for every path through the medium that contributes light, the path $\hat{\beta}$ must be restricted to

$$\vec{x}-\vec{x}' = \int_0^s \hat{\beta}(s')ds'$$

When we take the magnitude of this expression and use the Schwarz inequality

$$\begin{aligned}
|\vec{x}-\vec{x}'| &= \left|\int_0^s \hat{\beta}(s')ds'\right| \\
&\leq \int_0^s |\hat{\beta}(s')|\, ds' \\
&\leq s
\end{aligned}$$

So it must be exactly true that $G_{MFS} = 0$ whenever $|\vec{x}-\vec{x}'| > s$. The procedure that we need to follow in our approximate evaluation must pay attention this constraint. For this purpose, we can arrange the representation of unit vectors $\hat{\beta}$ in terms of a component perpendicular to $\vec{x}-\vec{x}'$, and one parallel to it. When we label the perpendicular component by the two dimesional vector $\vec{\gamma}$, the representation is

$$\hat{\beta} = \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|}\sqrt{1-|\vec{\gamma}|^2}+\vec{\gamma} \qquad (7)$$

with the understanding that $\vec{\gamma}\cdot(\vec{x}-\vec{x}') = 0$. Then the delta function spatial constraint is divided into two constraints:

$$|\vec{x}-\vec{x}'| = \int_0^s \sqrt{1-|\vec{\gamma}(s')|^2}\, ds' \qquad (8)$$

$$0 = \int_0^s \vec{\gamma}(s')\, ds' \qquad (9)$$

The benefit of this expansion using $\vec{\gamma}$ is that we can enforce the causality constraint. However, it does not describe paths with segments that are backwards facing relative to $\vec{x}-\vec{x}'$, which is the reason why the small angle approximation does a poor job with large angles in the radiance distribution. Fortunately, the large angle behavior has been handled separately, and we are only interested here in multiple forward scattering.

In the multiple forward scattering scenario, we anticipate that all paths are deviations from the forward direction, we can use the deviation vector $\vec{\gamma}$ as a small quantity in the first constraint. To leading order then, the time delay satisfies

$$s = |\vec{x} - \vec{x}'| + \frac{1}{2}\int_0^s |\vec{\gamma}(s')|^2 \, ds' \tag{10}$$

This order of expansion is a reasonable one to use because it enforces $s \geq |\vec{x} - \vec{x}'|$ no matter the deviation path chosen. Iterating the expansion, we obtain an explicit expression for s as:

$$s = |\vec{x} - \vec{x}'| + \frac{1}{2}\int_0^{|\vec{x}-\vec{x}'|} |\vec{\gamma}(s')|^2 \, ds' \tag{11}$$

Now the procedure is to go back to the path integral expression for G_{MFS} and substitute for s either equation 11 or the simpler expression $s = |\vec{x} - \vec{x}'|$ into each term, depending on the order of the term, so that all dependence up to quadratic in $\vec{\gamma}$ is kept, but higher orders are ignored. Using the shorthand $|\vec{x} - \vec{x}'| \equiv z$, $\langle\theta^2\rangle^2 b / \left(1 - e^{-bz}\right) \equiv T$, and $1/(aT) \equiv L^2$, the resulting path integral is

$$\begin{aligned} G_{MFS} &= e^{-az}\left(1 - e^{-bz}\right) \\ &\times \int d\mu(\vec{\gamma})\, \delta\left(s - z - \frac{1}{2}\int_0^s ds' \, \left|\vec{\gamma}(s')\right|^2\right) \\ &\times \delta\left(\vec{\gamma}(0) - \vec{\gamma}_{\hat{n}'}\right)\, \delta\left(\vec{\gamma}(z) - \vec{\gamma}_{\hat{n}}\right)\, \delta\left(\int_0^z \vec{\gamma}(s')\, ds'\right) \\ &\times \exp\left(-\frac{1}{2T}\int_0^z ds' \left\{ \left|\frac{d\vec{\gamma}(s')}{ds'}\right|^2 + \frac{|\vec{\gamma}(s')|^2}{L^2} \right\}\right) \end{aligned}$$

and $\vec{\gamma}_{\hat{n}'}$, $\vec{\gamma}_{\hat{n}}$ are the components of $\hat{n}'$ and $\hat{n}$ that are perpendicular to $\vec{x} - \vec{x}'$.

In the original definition of the deviation vector $\vec{\gamma}$, it is bounded to a unit disk, i.e. $|\vec{\gamma}|^2 \leq 1$. However, our approximations have produced an expression for the path integral that is mathematically well behaved if the support of $\vec{\gamma}$ were extended to the entire two dimensional plane R^2. Understanding that the integration over the additional area of the plane contributes relatively little error because larger values of $|\vec{\gamma}|^2$ are suppressed by the gaussian character of the integrand, the extended path integral has an exact evaluation into:

$$\begin{aligned} G_{MFS}(s, \vec{x}, \hat{n}, \vec{x}', \hat{n}') &= \frac{e^{-az}\left(1 - e^{-bz}\right)}{2\pi T^2 L^4 \, h(z/L)\, \sinh(z/L)} \\ &\times \exp\left(-M(z/L, \vec{\gamma}_{\hat{n}}, \vec{\gamma}_{\hat{n}'})\right) \\ &\times \delta\left(s - z - \frac{1}{2}\int_0^s ds' \, \left|\vec{\gamma}_{MFS}(s')\right|^2\right) \end{aligned}$$

with the functions defined as

$$h(x) = \frac{x\sinh(x) + 2(1 - \cosh(x))}{\sinh(x)} \tag{12}$$

$$\begin{aligned} M(x, \vec{\gamma}, \vec{\gamma}') &= \frac{(|\vec{\gamma}|^2 + |\vec{\gamma}'|^2)\sinh(2x)/2 \; - 2\vec{\gamma}\cdot\vec{\gamma}'\sinh(x)}{2TL\sinh^2(x)} \\ &+ \frac{|\vec{\gamma} + \vec{\gamma}'|^2(\cosh(x) - 1)^2}{2TL\sinh^2(x)\, h(x)} \end{aligned} \tag{13}$$

The multiple forward scattered path $\vec{\gamma}_{MFS}(s')$ is the path through the medium with the least attenuation between the points $(\vec{x}', \hat{n}')$ and $(\vec{x}, \hat{n})$

$$\vec{\gamma}_{MFS}(s') = \vec{\gamma}_{\hat{n}} f(s') + \vec{\gamma}_{\hat{n}'} f(s - s') \tag{14}$$

with the function

$$f(s') = \frac{\sinh(s'/L)}{\sinh(z/L)} \tag{15}$$

4.1 Angular and Spatial Blurring

Multiple scattering has the effect of blurring the radiance distribution, in both the angular and spatial degrees of freedom. In this section the amount of blurring is quantified direction from the solution in equation 13.

The first term of $M(x, \vec{\gamma}, \vec{\gamma}')$ is the angular blurring factor. To see more clearly how it acts as an angular spread, we can look at its form in the limit $z \ll L$. In this limit, the term is approximately

$$\frac{|\vec{\gamma} - \vec{\gamma}'|^2}{2Tz^2/L}$$

and it is reasonable to interpret $z\sqrt{T/L}$ as the angular spread of the light distribution over short distances. At large distances $z \gg L$, the blurring looks like

$$\frac{|\vec{\gamma}|^2 + |\vec{\gamma}'|^2}{2TL}$$

and the angular spread of $\sqrt{TL}$ is much narrower at large distances than would have been expected from the short distance spreading.

Similarly, the second term in M accounts for spatial spreading in the plane perpendicular to the primary direction $\vec{x} - \vec{x}'$. It arises from the process of exactly evaluating the spatial propagation contraint

$$\delta\left(\int_0^z \vec{\gamma}(s')\, ds'\right)$$

and has the spatial width

$$\sqrt{TLh(x)}L\frac{\sinh(x)}{\cosh(x) - 1}$$

There is also spatial spreading along $\vec{x} - \vec{x}'$ due to the constraint term

$$\delta\left(s - z - \frac{1}{2}\int_0^s ds' \, \left|\vec{\gamma}_{MFS}(s')\right|^2\right) \tag{16}$$

which delays the constributions from various times s according to the direction the light takes coming into the point.

For the multiple scattered Green's function in equation 3, the constraint is

$$s - s' = |\vec{x} - \vec{x}' - \hat{n}'s'| + \frac{1}{2}\int_0^{|\vec{x}-\vec{x}'-\hat{n}'s'|} ds'' |\vec{\gamma}_{MFS}(s'')|^2$$

To first order, the value of s' that satisfies this equation is

$$s' = \frac{s^2 - |\vec{x} - \vec{x}'|^2}{s - \hat{n}' \cdot (\vec{x} - \vec{x}')}$$

Since s' must be positive, this solution requires that $s \geq |\vec{x} - \vec{x}'|$. This allows us to evaluate the time integral in equation 3 to leave

$$\begin{aligned} \Delta G_{MS}(s, \vec{x}, \hat{n}, \vec{x}', \hat{n}') &= \theta\left(s - |\vec{x} - \vec{x}'|\right)\, b\left(\vec{x}' + \hat{n}'s'\right) \\ &\times T\left(0, s', \vec{x}' + \hat{n}'s', \hat{n}'\right) \\ &\times \frac{e^{-az}\left(1 - e^{-bz}\right)}{2\pi T^2 L^4 \, h(z/L)\, \sinh(z/L)} \\ &\times \int d\Omega'' \, \exp\left(-M(z/L, \vec{\gamma}_{\hat{n}}, \vec{\gamma}_{\hat{n}''})\right) \\ &\times P(\hat{n}'', \hat{n}') \end{aligned} \tag{17}$$

and the $\theta(x)$ is the Heaviside step function, which enforces the speed of light constraint.

This expression is the last ingredient needed for an implementable algorithm.

5 Conclusions

Further effort to implement this algorithm is in progress.

The small angle approximation is a very different regime of light propagation from the diffusion approximation used in models such as subsurface scattering. As such it should provide an additional useful tool in volume rendering for achieving a different kind of appearance for clouds, tissues, and water.

The structure of the Green's function makes the algorithm very flexible for introducing objects in the volume. In fact, although the multiple forward scattered portion of the result in equation 17 was built explicitly for a homogeneous medium, it applies with only small modification when the medium has internal structure. The essential ingredient is to define the total number of scattering lengths $\ell(s)$ along a path by

$$\frac{d\ell(s')}{ds'} = b(\vec{x}' + \hat{n}'s')$$

With this definition, all of the dependence on s and b can be absorbed into ℓ. A ray march to evaluate the radiance marches out fixed steps in ℓ, and the derived steps in s are variable depending on the variability of the medium along the ray march.

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