A Note on Representing Multiple Scatter as Multiple Internal Lights

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1 Introduction

This is a very brief note that outlines the derivation of a connection between a heuristic, art-driven fake multiple scattering method with a mathematically/physically motivated breakdown of radiative transfer that expresses a similar outlook. This note is very brief, providing just an outline for the approach. More details need to be filled in through numerical implementation.

2 Radiative Transfer via Kernel Integration

The radiative transfer equation has a solution in terms of a propagation kernel G:

$$L(\mathbf{x}, \hat{\mathbf{n}}) \;=\; \int_0^\infty ds \int d^3x' \int d\Omega' \; G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \; S(\mathbf{x}', \hat{\mathbf{n}}') \tag{1}$$

where S is the distribution of light emitted by an "external light source". The kernel satisfies the time-dependent radiative transfer equation

$$\left\{ \frac{\partial}{\partial s} \;+\; \hat{\mathbf{n}} \cdot \nabla \;+\; c(\mathbf{x}) \right\} G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \;=\; b(\mathbf{x}) \int d\Omega'' \; P(\hat{\mathbf{n}}, \hat{\mathbf{n}}'') \, G(s, \mathbf{x}, \hat{\mathbf{n}}'', \mathbf{x}', \hat{\mathbf{n}}') \tag{2}$$

along with the initial condition

$$G(s = 0, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \;=\; \delta(\mathbf{x} - \mathbf{x}') \; \delta(\hat{\mathbf{n}} - \hat{\mathbf{n}}') \tag{3}$$

Note that one way to compute approximations for the kernel is to apply a perturbation approach in the number of scattering events. The single-scatter approximation is implemented numerically as a ray march process, with variable s serving as the arc-length of the path of the march, and the integral over s driving the ray march.

3 Path Length Segmentation

Before applying approximations, there is an alternative segmentation of the integral over path length than exploits the property of the kernel that it can be deconstructed in path length, i.e. for any length of path $s^* < s$,

$$G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') = \int d^3x'' \int d\Omega'' \; G(s - s^*, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}'', \hat{\mathbf{n}}'') \; G(s^*, \mathbf{x}'', \hat{\mathbf{n}}'', \mathbf{x}', \hat{\mathbf{n}}') \tag{4}$$

Using this exact relationship, the integral solution for radiance can be exactly expressed as two terms:

$$\begin{aligned} L(\mathbf{x}, \hat{\mathbf{n}}) &= \int_0^{s*} ds \int d^3x' \int d\Omega' \; G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \; S(\mathbf{x}', \hat{\mathbf{n}}') \\ &+ \int_0^{\infty} ds \int d^3x' \int d\Omega' \; G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \; F(\mathbf{x}', \hat{\mathbf{n}}') \end{aligned} \tag{5}$$

where F is an "internal light source" derived from propagating the external light source through the volume for the path length s^*:

$$F(\mathbf{x}, \hat{\mathbf{n}}) = \int d^3x' \int d\Omega' \; G(s^*, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \; S(\mathbf{x}', \hat{\mathbf{n}}') \tag{6}$$

A natural choice for the path length s^* is the longest distance through the volume that single-scattering would encounter. Then the first term on the right hand side is can be evaluated with the single-scatter approximation, which is the ray march process that is well known. The second term involves and additional "level of indirection" involving some amount of scattering from the light source into the volume, followed by scattering to the radiance location and direction. An interesting first idea for evaluating the second term is to again apply the single scattering approximation for the kernel in the second term, and a higher order multiple scatter expression for $G(s^*)$ in the expression for F. This would make the connection with heuristic artistic techniques of using internal lights, but now the internal light has a quantitatively precise form in F.

4 Monte Carlo Evaluation Using Internal Lights

For numerical evaluation, the spatial integral in the second term of equation 5 could be evaluated using a Monte Carlo procedure. Suppose $\rho(\mathbf{x})$ is the density field for the volume. The density could be used as the unnormalized probablity density for generating the spatial locations for the Monte Carlo evaluations. Choosing to approximate the spatial integral using M random point in space $\{\mathbf{x}_i, \; i = 1, \ldots, M\}$ distributed using the density, the second term becomes

$$\left(\int d^3x' \; \rho(\mathbf{x}') \right) \; \frac{1}{M} \sum_{i=1}^{M} \int_0^{\infty} ds \; \int d\Omega' \; G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}_i, \hat{\mathbf{n}}') \; \frac{F(\mathbf{x}_i, \hat{\mathbf{n}}')}{\rho(\mathbf{x}_i)} \tag{7}$$

This form for the second term is simply a ray marching process with M internal point lights

$$\left(\int d^3x' \ \rho(\mathbf{x}')\right) \ \frac{F(\mathbf{x}_i, \hat{\mathbf{n}}')}{M \ \rho(\mathbf{x}_i)} \tag{8}$$

5 Iteration

Using a more brief *operator notation*, equation 1 is written as

$$L \ = \ \int_0^\infty ds \ G(s) \ S \tag{9}$$

where the production of G times S implicitly includes convolution over space and angle. The path length segmentation result in equation 5 now looks like

$$L \ = \ \int_0^{s^*} ds \ G(s) \ S \ + \ \int_0^\infty ds \ G(s) \ G(s^*) \ S \tag{10}$$

Note that the segmentation process can be applied to the second term to give

$$L \ = \ \int_0^{s^*} ds \ G(s) \ S \ + \ \int_0^{s^*} ds \ G(s) \ G(s^*) \ S \ + \ \int_0^\infty ds \ G(s) \ G(2s^*) \ S \tag{11}$$

In this expression, we have used the fact that $G(s^*)G(s^*) = G(2s^*)$. This expansion can be iterated indefinitely, to give

$$L \ = \ \int_0^{s^*} ds \ G(s) \ \left(\sum_{n=0}^\infty G(ns^*)\right) \ S \tag{12}$$

Defining the kernel H as[1]

$$H(\mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \ = \ \sum_{n=0}^\infty G(ns^*, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \tag{13}$$

The full radiative transfer solution now looks like

$$L(\mathbf{x}, \hat{\mathbf{n}}) \ = \ \int_0^{s*} ds \ \int d^3x' \ \int d\Omega' \ G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \ J(\mathbf{x}', \hat{\mathbf{n}}') \tag{14}$$

where the "source" is

$$J(\mathbf{x}, \hat{\mathbf{n}}) \ = \ \int d^3x' \ \int d\Omega' \ H(\mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \ S(\mathbf{x}', \hat{\mathbf{n}}') \tag{15}$$

Similar to the previous discussion, we can use spatial Monte Carlo sampling to reduce this to

$$L(\mathbf{x}, \hat{\mathbf{n}}) \ = \ \frac{\int d^3x' \ \rho(\mathbf{x}')}{M} \sum_{i=1}^M \int_0^{s^*} ds \ \int d\Omega' \ G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}_i, \hat{\mathbf{n}}') \ \frac{J(\mathbf{x}_i, \hat{\mathbf{n}}')}{\rho(\mathbf{x}_i)} \tag{16}$$

[1] Note that, because the RTE is a linear integro-differential equation, the infinite sum can be evaluated in operator notation as $H = \sum_{n=0}^\infty G(ns^*) = (1 - G(s^*))^{-1}$.

This form of the radiance has several very desireable features. First, as written it does not approximate the scattering behavior, although to implement it in a renderer some approximation will have to be constructed for H. Second, we are free to choose the integration distance s^* so that it is convenient, i.e. small enough that $G(s)$ in equation 16 is reliably expressed as a single scatter kernel. This effectively makes ray marching an accurate numerical means of evaluating equation 16, with multiple faux lights $J(\mathbf{x}_i, \hat{\mathbf{n}}')/\rho(\mathbf{x}_i)$. Third, all of the compromises for numerical expedience occur in only two places: the Monte Carlo distribution of points in space, and the approximation that will have to be chosen for H, and consequently for J.

Regardless of the approximation scheme for J and the number of Monte Carlo points, equation 16 is a systematic framework for attacking practical computation of multiple scattering.

6 Single Scatter Ray March

Here we apply the single scatter expression for G in equation 16. This means we assume that we can effectively choose s^* to be small enough for single scatter to be appropriate.

For single scatter, the kernel is

$$\begin{aligned} G_{SS}(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') &= T(s, \mathbf{x}, \hat{\mathbf{n}})\ \delta(\hat{\mathbf{n}} - \hat{\mathbf{n}}')\ \delta(\mathbf{x} - \mathbf{x}' - \hat{\mathbf{n}}s) \\ &+ \int_0^s ds'\ T(s - s', \mathbf{x}, \hat{\mathbf{n}})\ \beta(\mathbf{x} - \hat{\mathbf{n}}(s - s'), \hat{\mathbf{n}}, \hat{\mathbf{n}}') \\ &\times\ \delta\left(\mathbf{x} - \mathbf{x}' - \hat{\mathbf{n}}'s' - \hat{\mathbf{n}}(s - s')\right)\ T\left(s', \mathbf{x} - \hat{\mathbf{n}}(s - s'), \hat{\mathbf{n}}'\right) \end{aligned} \quad (17)$$

where $\beta(\mathbf{x}, \hat{\mathbf{n}}, \hat{\mathbf{n}}')$ is the product of the spatially-varying scattering coefficient and the spatially-varying phase function:

$$\beta(\mathbf{x}, \hat{\mathbf{n}}, \hat{\mathbf{n}}') = b(\mathbf{x})\ P(\mathbf{x}, \hat{\mathbf{n}}, \hat{\mathbf{n}}') \quad (18)$$

and

$$T(s, \mathbf{x}, \hat{\mathbf{n}}) = \exp\left(-\int_0^s ds'\ c\left(\mathbf{x} - \hat{\mathbf{n}}(s - s')\right)\right) \quad (19)$$

and $c(\mathbf{x})$ is the spatially-varying extinction coefficient.

The two terms in the single scattering kernel divide up the radiance output into two terms. For the first term the two delta functions mean that we do not have to resort to Monte Carlo integration. Instead, we can apply the first single scattering term to equation 14, to get

$$L_0(\mathbf{x}, \hat{\mathbf{n}}) = \int_0^{s*} ds\ T(s, \mathbf{x}, \hat{\mathbf{n}})\ J(\mathbf{x} - \hat{\mathbf{n}}s, \hat{\mathbf{n}}) \quad (20)$$

For the second term there is only one delta function constraint, but it can be used to specify $\hat{\mathbf{n}}'$ and s' in terms of the other variables:

$$s' = \frac{|\mathbf{x} - \mathbf{x}_i - \hat{\mathbf{n}}s|^2}{2\left(s - \hat{\mathbf{n}} \cdot (\mathbf{x} - \mathbf{x}_i)\right)} \quad (21)$$

$$\hat{\mathbf{n}}' = \frac{\mathbf{x} - \mathbf{x}_i - \hat{\mathbf{n}}(s - s')}{s'} \tag{22}$$

The derivation of these expressions also makes it clear that there is a non-zero contribution only when $s > |\mathbf{x} - \mathbf{x}_i|$ and $s > \hat{\mathbf{n}} \cdot (\mathbf{x} - \mathbf{x}_i)$.

With these, the second term for the radiance is

$$\begin{aligned} L_1(\mathbf{x}, \hat{\mathbf{n}}) &= \frac{\int d^3x' \; \rho(\mathbf{x}_i)}{M} \sum_{i=1}^{M} \int_0^{s^*} ds \; \frac{T(s - s', \mathbf{x}, \hat{\mathbf{n}})}{s'^2} \; \frac{\beta(\mathbf{x} - \hat{\mathbf{n}}(s - s'), \hat{\mathbf{n}}, \hat{\mathbf{n}}')}{\rho(\mathbf{x}_i)} \\ &\times \; T\left(s', \mathbf{x} - \hat{\mathbf{n}}(s - s'), \hat{\mathbf{n}}'\right) \; J(\mathbf{x}_i, \hat{\mathbf{n}}')) \\ &\times \; \Theta\left(s - |\mathbf{x} - \mathbf{x}_i|\right) \\ &\times \; \Theta\left(s - \hat{\mathbf{n}} \cdot (\mathbf{x} - \mathbf{x}_i)\right) \end{aligned} \tag{23}$$

The last two factors in this expression are Heaviside step functions to insure the constraints.