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MEAN-FIELD BURGERS' MODEL OF TURBULENCE*

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ABSTRACT

A mean-field expansion of the Martin, Siggia, and Rose functional formalism for turbulence is proposed as a ''strong-coupling'' approximation. To carry out the expansion, the volume of wavenumber space must be truncated to a large but finite amount. Burgers' model with homogeneous average flow is examined in detail to several orders of approximation. The two point cumulant is calculated. The mean-field expansion is described as a perturbation expansion, with its small parameter inversely proportional to the truncated volume of wavenumber space. Renormalization and some features of the expansion are explored.

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I. INTRODUCTION

Attempts to solve the closure problem in the statistical theory of turbulence can generally be categorized in two ways. The first is a direct attempt to close the equations for the velocity moments using a relationship between a high-order moment and lower-order ones. Such schemes as Wyld diagrams[1], the Wiener-Hermite expansion[2], the Direct Interaction approximation[3], and the Eddy Damped Quasinormal Markovian[4] approximation have been very useful for gaining some understanding of the turbulent process[5], but do not as yet qualify as complete theories of turbulence because (among other reasons) the effects of higher orders in their expansions are unknown. These closure approximations are ''weak-coupling'' expansions in which some contributions from the nonlinear interaction are ignored, but which may be as large or larger than those used. In the case of Wyld diagrams, for example, the approximation is an expansion in powers of the Reynolds number, which has a very large value in turbulent flows.[6]

In the second category the approach is to circumvent the closure problem by constructing a generating, or characteristic functional for the averaged velocity moments. This was originally suggested by Hopf[7], and the formalism of Martin, Siggia, and Rose[8] (MSR) was used in some calculations. Martin and DeDominicis[9] calculated the energy spectrum of a stochastically stirred fluid, and found the spectrum depended strongly on the spectrum of the driving force. Their results seem useful for understanding the large scales of a turbulent flow[10], but it is unclear whether the small scales are adequately for.[11] As

with the closure approximations, Martin and DeDominicis assumed a ''weak-coupling'' expansion, but used Renormalization Group techniques to sum the expansion, including all the interaction terms ignored in the closure methods.

Several new approaches not based on either of these two categories have been suggested[12] which may become very useful, but will not be discussed here.

It is possible that the difficulty in both the closure and generating functional methods used to date is that a weak-coupling expansion -- even a completely summed expansion -- is inadequate to represent the essentially ''strongly-coupled'' nature of turbulence, in which the nonlinear interaction is as large as any other contribution to the flow. In the sections following, a method is presented of approximating the generating functional in the MSR formalism in order to emphasize the nonlinear terms in the equations of motion. In order to illustrate the method, simplify calculations and make the results as clear as possible, Burgers' nonlinear diffusion equation[13] is used. In essence, the approach is to evaluate exactly as many of the functional integrals in the MSR formalism as possible, and approximate the remaining integral in a mean-field or steepest descent expansion. This will introduce a length scale which can be specified in terms of ''measurable quantities'' by fixing its value at some initial time. The approximation also requires that the total volume of spatial- and temporal- wavenumber space be kept finite, although large enough to contain all the dynamics of the problem. Its size is determined by

renormalization in a way identical to fixing the length scale discussed above.

In section II the MSR formalism is discussed and the expansion of the generating functional in the mean-field is presented. This expansion is equivalent to a loop expansion in quantum field theory. In section III the mean-field expansion for the case of homogeneous average velocity is used to obtain the energy spectrum in the largest order of approximation, and to show that higher terms in the expansion correspond to a perturbation expansion in a dimensionless parameter g . The renormalization structure is examined in section IV, and it is shown that the parameter g is proportional to the square of the ratio of the smallest to some larger scale in the flow. Several features of the method are discussed in section V.

II. MSR FORMALISM AND THE MEAN-FIELD EXPANSION

The (one-dimensional) fluid velocity $U(x,t)$ can be made a stochastic quantity by driving the flow with a gaussian stochastic force $f(x,t)$. Burgers' equation in this situation is

$$\frac{\partial}{\partial t} U(x,t) - \nu \frac{\partial^2}{\partial x^2} U(x,t) + \frac{\partial}{\partial x}\left(\frac{1}{2} U^2(x,t)\right) = f(x,t) \quad . \tag{1}$$

The solution to this equation can be constructed when the flow field at an initial time is specified.[14] It is not the solution of (1) that is of interest however, but the average of the solution over the ensemble

of driving forces f . Any initial conditions should be applied to the averaged quantities.

The average of a functional $A(f)$ over the ensemble of f is defined as

$$\langle A\rangle = N^{-1}\int[df]\, A(f)\, \exp\{-\tfrac{1}{2}\int fc^{-1}f\} \quad ,$$

where

$$\int[df] = \prod_{x,t}\int_{-\infty}^{+\infty} df(x,t) \quad ;$$

N^{-1} is defined by $\langle 1\rangle = 1$; and $\langle ff\rangle = c$ is the only nonvanishing cumulant of the force f . The generating functional $Z(J)$ in the MSR formalism is

$$Z(J) = e^{iW(J)} = N^{-1}\int[df]\, \exp\{-\tfrac{1}{2}\int fc^{-1}f\} \times \int[dU][d\phi]\, \exp\{i\int JU\}\, \exp\{i\int\phi(G^{-1}U + UU' - f)\} \quad . \tag{2}$$

$G^{-1}_{x,t} = \frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial x^2}$ is the heat equation operator. The integral over the ϕ-field is a delta function allowing only solutions of (1) to be included in the integration over the velocity field U . $W(J)$ is the generator of cumulants, and $J(x,t)$ is an arbitrary field used to generate cumulants through the relation

$$\langle U(x_1,t_1)\ldots U(x_n,t_n)\rangle_c = (\tfrac{1}{i})^{N-1}\left\{\frac{\delta}{\delta J(x_1,t_1)}\cdots\frac{\delta}{\delta J(x_n,t_n)}\right\} W(J)\Big|_{J=0} \tag{3}$$

The U-integration is further restricted by the implicit

requirement that the average velocity satisfy an initial condition at time t_0:

$$\langle U(x,t)\rangle\Big|_{t=t_0} = U_0(x) \quad . \tag{4}$$

It will be shown that (4) can be satisfied by lifting this restriction on the U-integration and approximating the ϕ-integration by the mean-field expansion.

Evaluating the integral over the driving force $f(x,t)$,

$$Z(J) = N^{-1}\int[dU][d\phi]\, \exp\{-\tfrac{1}{2}\int\phi c\phi + i\int JU + i\int\phi(G^{-1}U + UU')\} \quad . \tag{5}$$

This is the form used by Martin and DeDominicis[9] to examine the isotropic Navier-Stokes problem. The term quadratic in the velocity field was assumed proportional to a coupling constant, and the energy spectrum was calculated by summing the weak-coupling expansion with Renormalization Group techniques. It is not necessary to assume a weak-coupling because the quadratic interaction allows the U-integration to be evaluated exactly. Doing so, (5) is replaced by

$$Z(J) = N^{-1}\int(d\phi)e^{iS(\phi)} \quad , \tag{6}$$

with

$$S(\phi) = \tfrac{1}{2}\int\phi\, c\,\phi + \tfrac{1}{2}\,\mathrm{Tr}\,\ln\phi' + \tfrac{1}{2}\int(J - \Delta^{-1}\phi)^2/\phi' \quad . \tag{7}$$

''Tr'' means sum over all values of x,t ; $\Delta^{-1}_{x,t} = \frac{\partial}{\partial t} + \nu \frac{\partial^2}{\partial x^2}$; and $\phi'(x,t) = \frac{\partial}{\partial x}\phi(x,t)$.

The remaining functional integral in (6) can only be approximated. The mean-field method approximates the ϕ-integration by a steepest-descent approximation, expanding $S(\phi)$ in a functional Taylor series about a mean-field $\bar{\phi}$, chosen to extremize $S(\phi)$. This technique has been used with great success to approximate generating functionals in quantum field theory for a variety of models.[15]

To construct the mean-field $\bar{\phi}$ from the extremal condition on $S(\phi)$, the term $\frac{1}{2}\mathrm{Tr}\,\ell n\phi'$ in (7) must be written as an integral over space and time variables. This can be done as follows: assume minimum length and time scales Δx and Δt respectively, which are much smaller than any length or time scale in the flow. Then

$$\mathrm{Tr}\,\ell n\phi' = \sum_{n,m=-\infty}^{+\infty} \ell n\phi'(n\Delta x, m\Delta t) = (\Delta x\Delta t)^{-1} \sum (\Delta x\Delta t)\,\ell n\phi'(n\Delta x, m\Delta t) \quad .$$

Since Δx , Δt are extremely small,

$$\sum \Delta x\Delta t\ \ell n\phi'(n\Delta x, m\Delta t) = \int dx dt\ \ell n\phi'(x,t),$$

and $(\Delta x\Delta t)^{-1} = \pi^2\delta(x=0)\delta(t=0)$, where $\delta(y)$ is the Dirac delta function. Using the fourier transform representation of $\delta(y)$ as $y \to 0$, $\pi^2\delta(x=0)\delta(t=0) = 1/4\int dk d\omega \equiv \Lambda$. The quantity Λ is the volume of spatial- and temporal- wavenumber space available to the flow. In order to carry out the mean-field expansion, Λ must be kept finite. It will be shown in section IV that Λ can be kept large enough to include the smallest as well as the largest scales of the flow. The wavenumber volume Λ also has the important role of determining the

size of corrections to any order of approximation, so that as long as Λ is large in a way which will be precisely stated, corrections will be small.

The functional $S(\phi)$ is

$$S(\phi) = \frac{i}{2}\int \phi c \phi + \frac{i\Lambda}{2}\int \ell n\phi' + \frac{1}{2}\int (J - \Delta^{-1}\phi)^2/\phi' \quad . \tag{8}$$

The mean-field $\bar{\phi}$ is defined by requiring that $(\delta S(\phi)/\delta\phi(x,t))\big|_{\phi=\bar{\phi}} = 0$. This extremal condition can be separated into two equations by introducing a second mean-field $\bar{V}$. The two mean-fields satisfy

$$G^{-1}_{x,t}\,\bar{V}_J(x,t) + \frac{\partial}{\partial x}\left(\frac{1}{2}\bar{V}^2_J(x,t)\right) = \frac{i\Lambda}{2}\frac{\partial}{\partial x}\left(\frac{1}{\bar{\phi}'_J(x,t)}\right) - i\int c(x,t;z,z_o)\,dz dz_o\,\bar{\phi}_J(z,z_o) \ , \tag{9}$$

$$\Delta^{-1}_{x,t}\,\bar{\phi}_J(x,t) + \bar{V}_J(x,t)\frac{\partial}{\partial x}\bar{\phi}_J(x,t) = J(x,t) \ . \tag{10}$$

Equation (9) is the extremal condition $\delta S/\delta\phi = 0$, and equation (10) is the definition of $\bar{V}_J$. The subscript ''J'' refers to the arbitrary field $J(x,t)$, which drives $\bar{\phi}_J(x,t)$.

The mean-field expansion is obtained by setting $\phi \to \bar{\phi}_J + \phi$ in (6), and using

$$S(\phi) \to S(\bar{\phi}_J) + \frac{1}{2}\int \phi\phi S^{(2)}(\bar{\phi}_J) + \frac{1}{3!}\int \phi\phi\phi S^{(3)}(\bar{\phi}_J) + \ldots \quad ,$$

where

$$S^{(n)}(\ _J)(x_1\ ,\ t_1\ ;\ \ldots\ ;\ x_n\ ,\ t_n) = \left(\frac{\delta^n S(\phi)}{\delta\phi(x_1\ ,\ t_1)\ldots\delta\phi(x_n,t_n)}\right)_{\phi\ =\ \bar{\phi}_J} \quad :$$

$$Z(J) = \exp\{iS(\bar{\phi}_J)\}\ N^{-1}\int(d\phi)\ \exp\{\frac{i}{2}\int\phi\phi S^{(2)}(\bar{\phi}_J) + \frac{i}{3!}\int\phi\phi\phi S^{(3)}(\bar{\phi}_J)\} \quad .$$

Keeping only terms out to order n=2 , the generator of cumulants is

$$W(J) = W^{(0)}(J) + W^{(1)}(J) - \ell nN\ , \tag{11}$$

where

$$W^{(0)}(J) \ =\ S(\bar{\phi}_J) = \frac{1}{2}\int\bar{\phi}_J c\bar{\phi}_J + \frac{1}{2}\,\mathrm{Tr}\ell n\bar{\phi}_J' + \frac{1}{2}\int\bar{V}_J^2\bar{\phi}_J'\ , \tag{12a}$$

$$W^{(1)}(J) \ =\ \frac{1}{2}\,\mathrm{Tr}\ell n\left|S^{(2)}(\bar{\phi}_J)\right|\ . \tag{12b}$$

This is the loop expansion commonly employed in quantum field theory. The diagrammatic representation of $W^{(n)}(J)$ contains only diagrams with n closed loops. In the next section it will be shown that $W^{(0)} + W^{(1)}$ gives the largest contribution to a cumulant; all higher loops are proportional to a power of the small parameter g .

In order for any approximation to be ''strongly-coupled'' the lowest order term must be of the same size as the nonlinear term in Burgers' equation, and indeed the last term $W^{(0)}(J)$ is of the same form as the nonlinear term in (5), if $\bar{V}_J(x,t)$ is interpreted as a fluid velocity. The average velocity is, from (3),

$$\langle U(x,t)\rangle = \frac{\delta}{\delta J(x,t)}\ W(J)\Big|_{J=0}\ ,$$

and in fact

$$\frac{\delta W^{(0)}(J)}{\delta J(x,t)} \ =\ \bar{V}_J(x,t)\ , \tag{13}$$

so that the average velocity $\langle U(x,t)\rangle = \bar{V}_0(x,t)$ to lowest order of approximation. To obtain (13), it is necessary to determine equations of motion for $(\delta\bar{V}_J(y,\tau)/\delta J(x,t))$, and $(\delta\bar{\phi}_J(y,\tau)/\delta_J(x,t))$ from the mean-field equations (9) and (10):

$$c^{-1}_{y,\tau}\frac{\bar{V}_J(y,\tau)}{J(x,t)} + \frac{\partial}{\partial y}\left(\bar{V}_J(y,\tau)\ \frac{\delta\bar{V}_J(y,\tau)}{\delta J(x,t)}\right) = -\frac{i\Lambda}{2}\frac{\partial}{\partial y}\left(\frac{1}{\bar{\phi}_J'(y,\tau)}\right)^2\frac{\partial}{\partial y}\ \frac{\delta\bar{\phi}_J(y,\tau)}{\delta J(x,t)}$$
$$-\,i\int c(y,\tau;z,z_o)dzdz_o\ \frac{\delta\bar{\phi}_J(z,z_o)}{\delta J(x,t)}\ , \tag{14}$$

$$\Delta^{-1}_{y,\tau}\frac{\delta\bar{\phi}_J(y,\tau)}{\delta J(x,t)} + \bar{V}_J(y,\tau)\frac{\partial}{\partial y}\frac{\delta\bar{\phi}_J(y,\tau)}{\delta J(x,t)} \ +\ \frac{\delta\bar{V}_J(y,\tau)}{\delta J(x,t)}\ \frac{\partial}{\partial y}\ \bar{\phi}_J(y,\tau) = \delta(x-y)\delta(t-\tau)\ . \tag{15}$$

In the lowest order of approximation it is clear how the initial condition (4) is to be satisfied: $\bar{V}_0(x,t)$ satisfies Burgers' equation driven by a complicated forcing, and can be made to satisfy (4). Including higher order terms need not change this because their contributions to $\langle U(x,t)\rangle$ will depend on $(\delta\bar{V}_J/\delta J)_{J=0}$ and $(\delta\bar{\phi}_J/\delta J)_{J=0}$, each of which can be made to vanish at the initial time. In the next section it will be shown that homogeneous flow has sufficient symmetry to restrict higher order terms to vanish identically, leaving $\langle U\rangle$ to be exactly equal to the lowest order expression.

Equations (9), (10), and (13) provide a complete lowest order scheme for calculating cumulants of arbitrarily many points. From (13), the n-point cumulant is

$$\langle U(x_1,t_1)\ldots U(x_n,t_n)\rangle_c = i^{1-n}\left\{\frac{\delta}{\delta J(x_2,t_2)}\cdots\frac{\delta}{\delta J(x_n,t_n)}\bar{V}_J(x_1,t_1)\right\}\Big|_{J=0} , \tag{16}$$

which can be obtained by differentiating (9) and (10) with respect to J a sufficient number of times, setting $J=0$, and solving the resulting linear partial differential equations. In general the last step is extremely difficult, although for the homogeneous case to be considered all the steps are straightforward.

To obtain higher order terms, the $S^{(n)}(\bar{\phi}_J)$ must be calculated. For $n=2$,

$$S^{(2)}(\bar{\phi}_J)(x,t;y,\tau) = [\delta^2 S(\phi)/\delta\phi(x,t)\delta\phi(y,\tau)]_{\phi=\bar{\phi}_J} =$$

$$-\left\{\bar{G}^{-1}_{x,t}(J)\frac{1}{\bar{\phi}'_J(x,t)}\bar{\Delta}^{-1}_{x,t}(J) - \frac{i\Lambda}{2}\frac{\partial}{\partial x}\left(\frac{1}{\bar{\phi}'_J(x,t)}\right)^2\frac{\partial}{\partial x}\right\}\delta(x-y)\delta(t-\tau) , \tag{17}$$

where

$$\bar{G}^{-1}_{x,t}(J) = G^{-1}_{x,t} + \frac{\partial}{\partial x}\bar{V}_J(x,t) ,$$

$$\bar{\Delta}^{-1}_{x,t}(J) = \Delta^{-1}_{x,t} + \bar{V}_J(x,t)\frac{\partial}{\partial x} .$$

This will be used in the next section to calculate contributions from $W^{(1)}(J)$.

III. HOMOGENEOUS FLOW

In the remaining sections a particular case will be studied. It will be assumed that the fluid exists in infinite space and time domains, the forcing spectrum $c=0$, and the average velocity is some constant value υ which could be zero. This is not a particularly interesting problem in the study of Burgers' equation because homogeneity eliminates the possibility of having shocks in the spectrum. It has been chosen for three reasons: (1) all quantities needed can be calculated without difficulty; (2) the absence of such features as shocks and ''large-scale'' objects allows a clearer illustration of the effects of the averaging process; and (3) this method should be applicable to the Navier-Stokes homogeneous turbulence problem, and it is hoped the results presented here could be compared with higher dimensional calculations. Shocks will be discussed briefly in section V, where their effect on the cumulant spectrum will be outlined.

If $\bar{V}_0(x,t) = \upsilon$, then setting $J=0$ in (9) and (10), $\bar{\phi}_0(x,t)$ must satisfy

$$\frac{\partial}{\partial x^2}\bar{\phi}_o(x,t) = 0$$

$$\Delta^{-1}_{x,t}\bar{\phi}_o(x,t) + \upsilon\frac{\partial}{\partial x}\bar{\phi}_o(x,t) = 0$$

From the first of these equations, $\bar{\phi}_0(x,t)\propto x$. Since $\bar{\phi}_0$ is invariant under the Galilean transformation $x \to x + \beta t$,

$\upsilon \to \upsilon + \beta$, the only solution is $\bar{\phi}_0(x,t) \propto x - \upsilon t$. The arbitrary coefficient in $\bar{\phi}_0$ is chosen as

$$\bar{\phi}_o(x,t) = \frac{-i\Lambda}{2}\left(\frac{\ell}{\nu}\right)^2 (x-\upsilon t) .$$

This choice will simplify the form of expressions to come. The parameter ℓ has dimensions of length and is arbitrary. Its value must be fixed by specifying initial conditions for averaged quantities in addition to (4), and is closely tied to the question of renormalization and the size of corrections. This will be discussed in section IV.

Setting J=0 , equations (14) and (15) are linear equations with constant coefficients Their solution in terms of fourier integrals is

$$\left(\frac{\delta\bar{V}_J(x,\tau)}{\delta J(x,t)}\right)_{J=0} = \frac{2i}{\Lambda}(\nu/\ell)^2 \int \frac{d^2k}{(2\pi)^2} H(k,\omega)E(k,\omega;y-x;\tau-t), \tag{18}$$

$$\left(\frac{\delta\bar{\phi}_J(y,\tau)}{\delta J(x,t)}\right)_{J=0} = \int \frac{d^2k}{(2\pi)^2} F(k,\omega)E(k,\omega;y-x;\tau-t) , \tag{19}$$

with $d^2k = dkd\omega$, $E(k,\omega;y-x;\tau-t) = \exp\{ik(y-x-\upsilon(\tau-t))+i\omega(\tau-t)\}$,

and

$$H(k,\omega) = (\nu/\ell)^2 \frac{k^2}{\omega^2+\nu^2k^4+\frac{\nu^2}{\ell^2}k^2} ,$$

$$F(k,) = \frac{-(i\omega+\nu k^2)}{\omega^2+\nu^2k^4+\frac{\nu^2}{\ell^2}k^2} .$$

The first two terms in the denominator of $H(k,\omega)$ or $F(k,\omega)$ are connected with viscous diffusion processes in the fluid. The last is

due to the nonlinear interaction. To lowest order approximation, the cumulant two point function is

$$\langle U(x_1,t_1)U(x_2,t_2)\rangle_c^{(0)} = \left(\frac{\nu}{\ell}\right)^2 \frac{1}{2\pi^2\Lambda}\int d^2k\, H(k,\omega)\, E(k,\omega;y-x;\tau-t) . \tag{20}$$

The contribution of the one-loop correction comes from differentiating (12b):

$$\langle U(x,t)\rangle = \upsilon + \frac{\delta}{\delta J(x,t)}\ \frac{i}{2}\,\mathrm{Tr}\ln|S^{(2)}(\bar{\phi}_J)|\Big|_{J=0} = \upsilon + \Delta\upsilon(x,t) ,$$

$$\langle U(x_1,t_1)U(x_2,t_2)\rangle_c = \langle U(x_1,t_1)U(x_2,t_2)\rangle_c^{(0)} + \frac{\delta}{\delta J(x_1,t_1)}\frac{\delta}{\delta J(x_2,t_2)}\frac{1}{2}\,\mathrm{Tr}\ln|S^{(2)}(\bar{\phi}_J)|\Big|_{J=0}$$
$$= \langle U(x_1,t_1)U(x_2,t_2)\rangle_c^{(0)} + \Delta\langle U(x_1,t_1)U(x_2,t_2)\rangle_c . \tag{21}$$

Carrying out the first differentiation, the average velocity is left unchanged because of the homogeneity assumption. This can be shown in general in the following way: assume $W(J)$ can be written as

$$W(J) = W^{(0)}(J) + \Delta W(J) . \tag{22}$$

For the purpose of this argument, it is assumed that $\Delta W(J)$ contains all the corrections to the lowest order approximation, and (22) is an exact statement. As can be seen in the loop expansion, $\Delta W(J)$ depends on J only implicitly through its dependence on $\bar{V}_J(x,t)$ and $\bar{\phi}'_J(x,t)$. Using the functional chain rule,

$$\Delta\upsilon(x,t) = \int dy d\tau \left\{ \left(\frac{\delta(\Delta W)}{\delta\bar{V}_J(y,\tau)}\right)_{J=0}\left(\frac{\delta\bar{V}_J(y,\tau)}{\delta J(x,t)}\right)_{J=0} + \left(\frac{\delta(\Delta W)}{\delta\bar{\phi}'_J(y,\tau)}\right)_{J=0}\frac{\partial}{\partial y}\left(\frac{\delta\bar{\phi}_J(y,\tau)}{\delta J(x,t)}\right)_{J=0}\right\} .$$

The quantities $(\delta(\Delta W)/\delta\bar{V}_J(y,\tau))_{J=0}$ and $(\delta(\Delta W)/\delta\bar{\phi}'_J(y,\tau))_{J=0}$ depend on the time variable τ only through their dependence on $\bar{V}_0(y,\tau)$ and $\bar{\phi}'_0(y,\tau)$, which in the homogeneous solution are time independent. From (18) and (19), all dependence on τ in the integral expression for $\Delta\nu$ is of the form $\tau-t$, which when the τ integral is evaluated is independent of t. Since the correction $\Delta\nu(x,t)$ is independent of time, the time t may be set to $\pm\infty$, where $(\delta\bar{V}_J/\delta J)_{J=0}$ and $(\delta\bar{\phi}_J/\delta J)_{J=0}$ vanish, forcing $\Delta\nu$ to vanish. The solution

$$\langle U(x,t)\rangle = \nu$$

is an exact solution to all orders in the mean-field expansion.

The one-loop correction to the two point function in (21) is

$$\Delta\langle U(x_1,t_1)U(x_2,t_2)\rangle_c = \frac{1}{2}\,\mathrm{Tr}\left\{S^{(2)}(\bar{\phi}_o)^{-1}\left(\frac{\delta^2 S^{(2)}(\bar{\phi}_J)}{\delta J(x_1,t_1)\,\delta J(x_2,t_2)}\right)_{J=0}\right\} - \frac{1}{2}\,\mathrm{Tr}\left\{S^{(2)-1}(\bar{\phi}_o)\left(\frac{\delta S^{(2)}(\bar{\phi}_J)}{\delta J(x_1,t_1)}\right)_{J=0} S^{(2)-1}(\bar{\phi}_o)\left(\frac{\delta S^{(2)}(\bar{\phi}_o)}{\delta J(x_2,t_2)}\right)_{J=0}\right\} . \tag{23}$$

Using (17), the first term of (23) will require knowledge of $(\delta^2\,\bar{V}_J(y,\tau)\,/\,\delta J(x_1,t_1)\delta J(x_2,t_2))_{J=0}$ and $(\delta^2\bar{\phi}_J(y,\tau)\,/\,\delta J(x_1,t_1)\delta J(x_2,t_2))_{J=0}$.

These can be obtained by differentiating (14) and (15) and solving in terms of fourier transforms. The final result is

$$\Delta\langle U(x_1,t_1)U(x_2,t_2)\rangle_c = \left(\frac{\nu}{\ell}\right)^2(2\pi^2\Lambda)^{-1}\left\{\int d^2k H(k,\omega)E(k,\omega;x_1-x_2,t_1-t_2) + O(g)\right\} ,$$

where $g = \frac{1}{\Lambda}\int dk d\omega H(k,\omega)$. The first term of $\Delta\langle UU\rangle$ is $\langle U(x_1,t_1)U(x_2,t_2)\rangle_c^{(0)}$.

Two-loop corrections to the two point cumulant have been examined, and the largest terms are proportional to g . The pattern which emerges is that the largest contributions from the $(n+1)$st loop are proportional to g^n . If the dimensionless parameter g is small, then the mean-field approximation is a perturbation expansion in g . In the next section specific initial conditions will be determined which guarantee that $g \ll 1$.

The two-point cumulant after including the one-loop correction is

$$\langle U(x_1,t_1)U(x_2,t_2)\rangle_c = \left(\frac{\nu}{\ell}\right)^2(\pi^2\Lambda)^{-1}\left\{\int d^2k H(k,\omega)E(k,\omega;x_1-x_2,t_1-t_2) + O(g)\right\} . \tag{24}$$

Ignoring terms of order g and smaller, the integral in (24) over frequency ω may be evaluated:

$$\langle U(x_1,t_1)U(x_2,t_2)\rangle_c = \int_0^\infty dk E(k,t)\exp\{ik(x_1-x_2-\nu(t_1-t_2))\} , \tag{25}$$

where $t = |t_1-t_2|$, and

$$E(k,t) = \left(\frac{\nu}{\ell}\right)^2\frac{1}{\pi\Lambda}\,\frac{(\nu/\ell)^2k^2}{[\nu^2k^4+(\frac{\nu}{\ell})^2k^2]^{\frac{1}{2}}}\exp\{-t(\nu^2k^4+\frac{\nu^2}{\ell^2}k^2)^{\frac{1}{2}}\} \tag{26}$$

is the energy spectrum. Restricting the discussion to the case

$x_1 = x_2$, $\nu = 0$, it will be useful to evaluate the integral in (25) in dimensionless form by defining $s = k\ell$. Then

$$\langle U(x_1,t_1)U(x_1,t_1+t)\rangle_c = E_o \int_0^\infty ds \, \Sigma(s,r) \quad , \tag{27}$$

with $E_o = \frac{\nu^3}{\ell^5} \frac{1}{\pi\Lambda}$ and

$$\Sigma(s,r) = \frac{s^2}{[s^4+s^2]^{\frac{1}{2}}} \exp\{-\frac{1}{r}(s^4+s^2)^{\frac{1}{2}}\} .$$

The parameter $r = \frac{\ell^2}{\nu t}$ is a Reynolds number which determines the importance of the nonlinear and viscous terms in diffusing the velocity correlation through time. If $r \gg 1$ ($r \ll 1$) the viscous (nonlinear) terms dominate. This is a reverse of the standard picture that nonlinear terms are important when the Reynolds number is large. The difference is that in the standard picture the Reynolds number is defined in terms of large scale parameters, whereas r is defined in terms of the length scale ℓ , and is not directly related to large scale features. In the next section ℓ will be specified in terms of large scale parameters, so that $r \approx R^{-1}$ when the large scale Reynolds number $R \gg 1$.

The dimensionlesss spectrum $\Sigma(s,r)$ is plotted in figure 1 for several values of r . As $k \to \infty$, the spectrum is exponentially damped. There is no extended region of wavenumbers with the shock spectrum $E(k) \sim k^{-2}$ because homogeneity prevents large scale shocks from forming. This will be discussed in more detail in section V.

Using the change of variable $s = \sinh\theta/2$, the integral in (27) can be given a representation in terms of a Lommel function[16], and is

plotted in figure 2. Two distinct regions are clearly present[17], and are examined in detail in the next section.

IV. RENORMALIZATION

The two point cumulant in (25) or (27) depends on the truncated wavenumber volume Λ and the length parameter ℓ . Their values can be fixed in terms of measurable quantities in a renormalization procedure by specifying at an initial time separation t_0 the values of no more than two measurable quantities. If the two quantities were the dissipation ϵ and the rate of strain σ , then equation (25) or (27) would provide expressions for each of these in terms of Λ, ℓ, and t . At $t=t_0$, these can be inverted to express Λ and ℓ in terms of $t_0, \sigma_0, \epsilon_0$, giving a two point function which depends entirely on measurable quantities. This will be carried out in this section for the two limits $r \gg 1$ and $r \ll 1$. In each case an expression for g will be found and the condition $g \ll 1$ will be used to place an upper bound on the value of t_0 related to the Kolmogorov time microscale. An interpretation of the upper bound is that by measuring quantities initially at a very small time separation, the wavenumber volume is forced to be large enough to accommodate such small scales. The upper bounds to be found shortly guarantee that the wavenumber volume is sufficiently large to include the dissipation scales.

In addition to the dissipation and rate of strain, other measurable quantities can be used to renormalize Λ and ℓ . A list

of a few of them contains:

(i) Energy $(U'(t))^2 = \langle U(x_1,t_1)U(x_1,t_1+t)\rangle_c$

(ii) Dissipation $\varepsilon(t) = -\frac{d}{dt}U'^2(t)$

(iii) Rate of strain $\sigma^2(t) = \left\langle \frac{\partial U(x_1,t_1)}{\partial x_1}\frac{\partial U(x_1,t_1+t)}{\partial x_1}\right\rangle_c$

(iv) Integral length scale $L(t) = \int_0^\infty dk\, k^{-1}E(k,t) \Big/ \int_0^\infty dk E(k,t)$.

Other quantities can of course be used besides the ones listed here.

To examine the size of the perturbation parameter g , an explicit expression must be found for it in terms of Λ and ℓ . Suppose Λ is very large so that the boundary of the wavenumber volume is very far away from the small scale dynamics. Any physical processes at such large wavenumbers are viscous dissipation processes. If Δx and Δt are the minimum length and time scales repectively associated with the boundary, then they an be related to each other through the diffusive process: $\Delta t \sim (\Delta x)^2/\nu$. In the integral expression for g ,

$$g = \frac{1}{\Lambda}\int dk d\omega\, H(k,\omega) , \tag{28}$$

the change of variables $k = \rho\cos\theta$, $\omega = \nu\rho^2\sin\theta$ may be made. Then the integration limits are $0\le\theta\le 2\pi$ and $0\le\rho\le\frac{1}{\Delta x}$, so that

$$\Lambda \propto \nu/(\Delta x)^3 ,$$

and

$$g \propto \left(\frac{\nu}{\Lambda\ell^3}\right)^{2/3} . \tag{29}$$

This will be used to examine the condition $g \ll 1$.

<u>Case A</u>: $r \ll 1$.

The integral in (27) can be approximated by setting $s=rb$, and expanding in powers of r . The largest terms for each of (i) - (iv) are

$$U'^2(t) = \frac{1}{\pi\Lambda}\frac{\nu^3}{\ell^5}r^2(1+O(r)) = \frac{1}{\pi\Lambda}\left(\frac{\nu}{\ell}\right)t^{-2}$$

$$\varepsilon(t) = \frac{2}{\pi\Lambda}(\nu/\ell)t^{-3}$$

$$\sigma^2(t) = \frac{6}{\pi\Lambda}\nu^3/\ell^7 r^4 = \frac{6}{\pi\Lambda}\left(\frac{\ell}{\nu}\right)t^{-4}$$

$$L(t) = \ell/r = \nu t/\ell .$$

Choosing ε and σ to renormalize Λ, ℓ,

$$r = \frac{\ell^2}{\nu t} = r_0\left(\frac{t_0}{t}\right) ,$$

$$r_0 = \frac{\sigma_0^2}{3}\nu/\varepsilon_0 ,$$

and

$$g \propto \left(t_0^2\left(\frac{\varepsilon_0}{\nu}\right)\frac{1}{r_0}\right)^{2/3} .$$

For $g \ll 1$, the initial time separation t_0 has an upper bound $\left(r_0\left(\frac{\nu}{\varepsilon_0}\right)\right)^{1/2}$, which is smaller than the Kolmogorov time microscale.

Using the integral length $L(t)$ to renormalize ℓ , $\ell = \frac{\nu t_0}{L_0}$, and $r = R^{-1}(t_0/t)$, where $R = L_0^2/\nu t_0$ is a Reynolds number connected with large-scale motion.

Case B: $r \gg 1$

In case A the upper bound on the initial time separation was very small, suggesting that to renormalize Λ and ℓ a large r expansion should be used. The integral in (27) has a movable stationary point as $r \to \infty$, and Laplace's method can be used. The results are

$$U'^2(t) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\nu^{5/2}}{\pi\Lambda\ell^4} \frac{2}{e} t^{-1/2}\left(1 + \frac{\nu t}{2\ell^2}\right)$$

$$\varepsilon(t) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\nu^{5/2}}{\pi\Lambda\ell^4} \frac{1}{e} t^{-3/2}\left(1 - \frac{3}{2}\frac{\nu t}{\ell^2}\right)$$

$$\sigma^2(t) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\nu^{3/2}}{\pi\Lambda\ell^4} \frac{8}{3e^2} t^{-3/2}\left(1 - \frac{3}{2}\frac{\nu t}{\ell^2}\right)$$

$$L(t) \propto (\nu t)^{1/2} \quad .$$

As $t \to 0$, Burgers' equation (1) may be averaged directly to obtain $\varepsilon(t\to 0) = \sigma^2(t\to 0)$. The $r \gg 1$ approximation gives

$$\frac{\varepsilon(t)}{\nu\sigma^2(t)} = \frac{3e}{8} \simeq 1.02$$

as $t \to 0$. Solving for Λ and ℓ ,

$$r = r_0(t_0/t) \quad ,$$

with

$$r_0 = \left(\frac{8}{3e}\frac{\varepsilon_0}{\nu\sigma_0^2} - 1\right)^{-1} \gg 1$$

and

$$g \propto \left(t_0^2\left(\frac{\varepsilon_0}{\nu}\right) r_0^{\frac{1}{2}}\right)^{2/3} \quad .$$

The perturbation parameter g is small when the initial time

separation has upper bound $[r_0^{-\frac{1}{2}} (\nu/\varepsilon_0)]^{\frac{1}{2}}$, which is smaller than the Kolmogorov time microscale, as in Case A.

Notice that for $r \gg 1$, the first terms in $U'^2(t)$ and σ^2 are all proportional to $(\Lambda\ell^4)^{-1}$, and only one measurable quantity is needed to renormalize the expressions. At small time separations the transfer of information about the velocity field from one time to another is carried out by viscous diffusion. The important length scale is the diffusion scale $(\nu t)^{\frac{1}{2}}$, and the length scale ℓ is present only to ensure the correct dimensions of all objects calculated. As the time separation grows the transfer originates in the nonlinear interaction, r becomes small, and the cumulant decays according to case A.

V. FEATURES OF THE MEAN-FIELD EXPANSION

The picture of the dynamics which emerges from the mean-field expansion can be characterized by the two extreme ranges of r discussed in the last section. At small separations the dynamics is dominated by viscous diffusion and all quantities scale by a similarity transformation. When the separation becomes larger than ℓ , nonlinear diffusion controls the dynamics and similarity is lost. This is the sort of behavior described in cascade models, except that this picture runs from small to large scales and cascade models generally consider the transfer to be form large to small scales. The difference in the two pictures is a matter if viewpoint, however; the essential message

is the same: large scales are controlled by the nonlinear interaction, small scales by viscous diffusion. The mean-field expansion provides a length scale ℓ which controls the importance of the two processes.

What has not been seen in the mean-field expansion of Burgers' equation is the k^{-2} spectrum characteristic of shocks. The reason why is that the two point cumulant $\langle U(x_1,t_1)U(x_2,t_2)\rangle_c$ is generated by fluctuations of the average flow, and cannot alter the large scale structure. Since the average flow considered here contains no shocks, none can be present in the cumulant.

The effect of a simple shock can be seen using the results obtained so far. Consider an average flow with a simple discontinuity shock:

$$\langle U(x,t)\rangle = \begin{Bmatrix} c & \text{when} & x>0 \\ -c & \text{when} & x<0 \end{Bmatrix} .$$

Then the two point cumulant is still (25), with ν replaced by $\pm c$ in the appropriate region. The energy spectrum is now the original cumulant spectrum plus a k^{-2} spectrum from the square of the average velocity. As $k \to \infty$, the cumulant spectrum vanishes. This may be one of many differences between one and three dimensional turbulence: in one dimension the -2 inertial range exponent comes from the average flow, whereas in three dimensions the -5/3 exponent in the inertial range is expected to come from the fluctuations. Work has begun on the mean-field expansion of the three dimensional Navier-Stokes problem.

It will be interesting to see if the two-point cumulant has any sort of inertial range, and what its exponent might be.

The truncated wavenumber volume Λ plays a fundamental role in all aspects of the mean-field expansion. Its existence defines the mean-field equations (9) and (10), the size of the perturbation parameter g, and the renormalization approach. It is also closely tied to the averaging process, which can be characterized by the amount of detail remaining in the flow after the averaging. If sufficiently sensitive equipment is available, an experiment can measure in complete detail the fluctuations of the velocity, treating them as a complicated but non stochastic flow. Averaging treats as stochastic the finer motion below some length and time scales Δx and Δt, which specify the volume Λ. Measuring the complete detail corresponds to $\Lambda \to \infty$, whereupon all cumulants and corrections vanish, and the mean-field equations (9) and (10) reduce to $\bar{\phi}_0 = \infty$, $c=0$, and Burgers' equation without a stochastic force. In this limit the mean-field expansion correctly reduces to the non-stochastic problem. The volume Λ provides a measure of the detail which can be resolved in the flow after the averaging process.

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REFERENCES

(1) H.W. Wyld, Ann. Phys. 14,143(1961).

(2) W.C. Meecham and A. Siegal, Phys. Fluids 1,1178(1964).

(3) D.C. Leslie, Developments in the Theory of Turbulence, Clarendon Press (1973).

(4) See for example, J.D. Fournier and U. Frisch, Phys. Rev. A 17,747(1978).

(5) A review of some of these approaches is contained in the lectures by S.A. Orszag in Fluid Dynamics, Les Houches (1973A), edited by R. Balian and J-L. Penbe.

(6) R.H. Kraichnan, in Dynamics of Fluids and Plasmas, Academic Press, (1966).

(7) E. Hopf, J. Rational Mech.Anal.1,87(1952).

(8) P.C. Martin, E.D. Siggia, H.A. Rose, Phys. Rev. A8,423(1973).

(9) C. DeDominicis, P.C. Martin, Phys.Rev.A19,419(1979).

(10) P.C. Martin and C. DeDominicis, Suppl. Prog. Theor. Phys.64,108(1978).

(11) R.H. Kraichnan, Phys. Rev. A25,3281(1982).

(12) See for example, U. Frisch and R. Morf, Phys. Rev. A23 2373(1981).

(13) J.M. Burgers, Statistical Models and Turbulence. Edited by M. Rosenblatt and C. Van Atta (1972).

(14) In general the explicit solution of (1) for arbitrary forcing is not known. For an analytical approach, see H.M. Fried and J. Tessendorf, J. Math Phys. (to appear). For the case of periodic forcing, see D.T. Jeng and W.C. Meecham, Phys. Fluids 15,504(1972).

(15) See for example D.J. Gross and A. Neveu, Phys. Rev. D10,3235(1974); Phys. Rep. 23(1976); R. Rajaraman, Solitons and Instantons, North-Holland (1982).

(16) I.S. Gradshteyn and I.W. Ryzhik, Tables of Integrals Series and Products, Academic Press (1965), integral #3.547.1.

(17) I. Hosokawa and K. Yamamoto, Phys. Fluids 13,1683(1970) numerically calculate the two point function using Monte Carlo integration of a generating functional and the Hopf-Cole solution for time-development. The data displayed in their figure 1 exhibits a ''bend'' similar to that in figure 2 in this paper, but is not seen in the closure approximations displayed along with their data. This similarity cannot be taken too seriously since shocks form as large scale features in their calculations, but are absent here. Nevertheless the comparison is interesting and warrants further study.

FIGURE CAPTIONS

Figure 1: Dimensionless energy spectrum versus $s=k\ell$.

Figure 2: Two point cumulant at a single space point versus time separation t . $r = \ell^2/\nu t$.

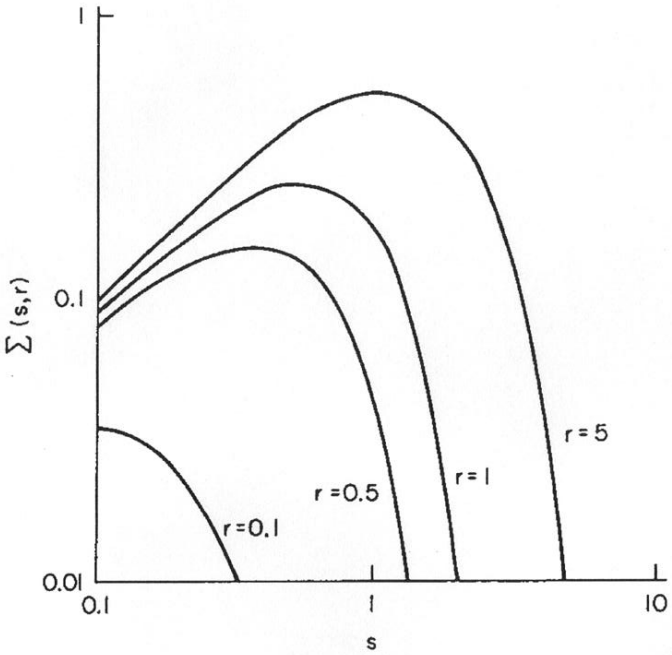

FIGURE 1

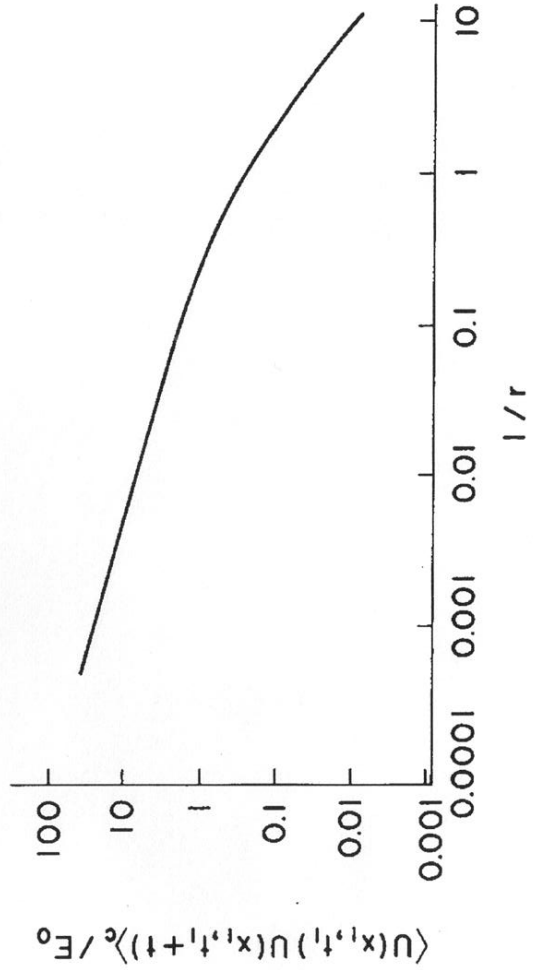

FIGURE 2