Green's functions at zero viscosity[a)]

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Fradkin-type propagator representations are written for solutions to Navier–Stokes and related equations, for arbitrary dimension D and arbitrary source geometry. In the limit of very small viscosity, velocity/vorticity solutions are given in terms of Cauchy position coordinates q of a particle advected by the velocity flow v, using a set of coupled equations for q and v. For localized point vortices in two dimensions, the vectors q become the time-dependent position coordinates of interacting vortices, and our equations reduce to those of the familiar, coupled vortex problem. The formalism is, however, able to discuss three-dimensional vortex motion, discrete or continuous, including the effects of vortex stretching. The mathematical structure of vortex stretching in a D-dimensional fluid without boundaries is conveniently described in terms of an SU(D) representation of these equations. Several simple examples are given in two dimensions, to anchor the method in the context of previously known, exact solutions. In three dimensions, vortex stretching effects are approximated using a previous "strong coupling" technique of particle physics, enabling one to build a crude model of the intermittent growth of enstrophy, which may signal the onset of turbulence. For isotropic turbulence, the possibility of a singularity in the inviscid enstrophy at a finite time is related to the behavior of a single function characterizing the intermittency.

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I. INTRODUCTION

Motivated by recent work suggesting chaotic behavior of a particle advected by the velocity flow due to three or more interacting vortices,[1] we would like to present a formalism based directly on the coupled velocity/vorticity equations of Navier–Stokes (NS) theory, which should be capable of describing situations involving specified, externally produced vortices as well as the possibility of spontaneous vortex generation at low viscosity. Based on this formalism, in this paper we offer an alternate way of characterizing known, two-dimensional flows, and a somewhat crude picture of the onset of turbulence in an infinite, three-dimensional, inviscid fluid without rigid boundaries.

To the best of our knowledge, the representations derived here are new; they are based upon a Green's function method invented by Fradkin[2] in the context of scattering problems in potential theory and quantum field theory. Fradkin's original functional differential forms may readily be converted to an equivalent functional integral formalism, and when applied to the NS problem it turns out that a great simplification can be made in the limit of small viscosity. There, the necessary functional integral can be well represented by an extremum method, analogous to the small $\hbar$ or large N expansions of quantum field theory, with corrections given in ascending powers of viscosity. We expect the spontaneous generation of vortices to be associated with such corrections; however, the general structure for arbitrary spatial dimension D and arbitrary source geometry and time dependence is most simply discussed in the limit of zero viscosity.

In this formalism, solutions to the inviscid NS problem are given in terms of a time-dependent position vector of a fictitious particle, or of a passive marker, whose position and velocity are codetermined by the exact velocity flow, in a construction which emerges from the extremum calculation appropriate to the small viscosity limit. For a system of distinct, point vortices, these marker coordinates represent the time- and position-dependent coordinates of the vortices themselves. An interesting feature of this method, which explicitly couples velocity and vorticity flow of the NS system in D dimensions, is the natural appearance of what may be an underlying SU(D) symmetry, associated in a nontrivial way with the presence of vortex stretching.

In this paper we present an alternative way of characterizing known, two-dimensional velocity/vortex flows, along with some simple generalizations, and a somewhat crude picture of what may be the onset of turbulence in an infinite, three-dimensional, inviscid fluid without rigid boundaries. This is easiest to see for the case of isotropic turbulence, where there is but one length scale characterizing the spatial scale of vortex motion. In the nonisotropic case where (at least) two length scales enter, there is another function which must be determined, and which acts to damp the intermittent growth, and the possibility of a finite-time singularity. These results, special to three dimensions, follow from a representation of vortex stretching effects in terms of the growth of a nonunitary SU(3) matrix, whose components are related to time integrals of velocity gradients. With the aid of a "strong-coupling" technique previously used in particle physics, one can obtain approximate forms valid for large velocity gradients. The nonlinearities of the problem are still formidable; but, in a crude, dimensional way, it is easy to watch the growth of vorticity, and, in an intermittent way, as a function which essentially controls the

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increase of enstrophy becomes large, not smoothly but in spurts.

The output of our method is closely related to solutions for vorticity suggested more than a century ago by Cauchy,[3] $\omega(t) = \omega_i(t_0)\partial/\partial a_i \mathbf{X}(\mathbf{a},t)$, $\mathbf{X}(\mathbf{a},t_0) = \mathbf{a}$, in terms of the position coordinates $\mathbf{X}(\mathbf{a},t)$ of the moving, material fluid; our "Marker" coordinates q are in essence Cauchy's $\mathbf{X}$, generalized to arbitrary vorticity source distributions.

II. THE FORMALISM

We begin with the standard NS equation

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\nabla^2\mathbf{v} = -\nabla p + \mathbf{f}(\mathbf{r},t), \tag{1}$$

where p denotes the fluid pressure, ν is the kinematic viscosity, and $\mathbf{f}$ represents a divergenceless velocity source. Taking the curl of (1) generates

$$\partial_t \omega + (\mathbf{v}\cdot\nabla)\omega - \nu\nabla^2\omega - (\omega\cdot\nabla)\mathbf{v} = \mathbf{g}(\mathbf{r},t), \tag{2}$$

where $\omega = \nabla\times\mathbf{v}$ is the vorticity and $\mathbf{g}$ denotes a corresponding divergenceless vorticity source, $\mathbf{g} = \nabla\times\mathbf{f}$. For a specified velocity field $\mathbf{v}(\mathbf{r},t)$, (2) is linear in ω, and can be solved in terms of an appropriate, if formal, functional representation, $\omega = \omega\{v\}$. That solution must then be combined with the relation $\omega = \nabla\times\mathbf{v}$, or its inverse $\mathbf{v} = \mathbf{v}_0 - \nabla^{-2}\nabla\times\omega$, where $\mathbf{v}_0$ is some specified, initial velocity field in the absence of $\mathbf{g}$ and ω, satisfying $\nabla\times\mathbf{v}_0 = 0$. Other methods of resolution of (2) are possible, but this is the simplest for our purposes; in fact, we shall further simplify matters by supposing that $\mathbf{v}_0$ denotes a constant, not-too-large velocity flow. A vorticity source of arbitrary strength $\mathbf{g}(\mathbf{r},t)$ is then turned on at $t = 0$, and we ask for the subsequent velocity/vorticity flow.

The corresponding solution to (2), for specified $\mathbf{v}(\mathbf{r},t)$ and under the assumption that ω vanishes for all $t<0$, is given by

$$\omega_a(\mathbf{r},t) = \int d^D y \int dy_0\, G_{ab}(\mathbf{r},\mathbf{y};t,y_0|v) g_b(\mathbf{y},y_0), \tag{3}$$

where $G[v]$ denotes that casual Green's function which satisfies[4]

$$\sum_{b=1}^{D} \{[\partial_t + (\mathbf{v}\cdot\nabla) - \nu\nabla^2]\delta_{ab} - \partial_b v_a\} G_{bc}(\mathbf{r},\mathbf{y};t,y_0|v)$$
$$= \delta^D(\mathbf{r}-\mathbf{y})\delta(t-y_0)\delta_{ac} \tag{4}$$

in D spatial dimensions. Using an obvious shorthand, one can write the formal equivalent of (4)

$$([\partial_t + (\mathbf{v}\cdot\nabla) - \nu\nabla^2] - (\partial v))G[v] = 1, \tag{5}$$

and a corresponding formal representation

$$G[v] = \int_0^\infty ds \exp\{-s([\partial_t + (\mathbf{v}\cdot\nabla) - \nu\nabla^2] - (\partial v))\}. \tag{6}$$

It will become clear, subsequently, that (6) does indeed define a retarded Green's function. The symbol $\mathbf{v}$ in (5) and (6) now denotes an operator with matrix elements diagonal in configuration space and time, $\langle \mathbf{x},x_0|\mathbf{v}|\mathbf{y},y_0\rangle = \mathbf{v}(\mathbf{x},x_0)\delta^D(\mathbf{x}-\mathbf{y})\delta(x_0-y_0)$, while ∂_+ and ∇ retain their customary operator meanings.

It is convenient to rewrite the square bracket of (6) as $[\partial_t - \nu(\nabla - \mathbf{v}/2\nu)^2 + \mathbf{v}^2/4\nu]$, since it is assumed that $\mathbf{v}$ is divergenceless, $\Sigma_a[\partial_a, v_a] = 0$; all the complications of the representation that follow arise because a general component of $\mathbf{v}(\mathbf{x},x_0)$ does not commute with ∇, or with ∇^2. With this step, one has

$$G[v] = \int_0^\infty ds \times\exp\Big\{-s\Big(\Big[\partial_t - \nu\Big(\nabla - \frac{\mathbf{v}}{2\nu}\Big)^2 + \frac{\mathbf{v}^2}{4\nu}\Big] - (\partial v)\Big)\Big\}. \tag{7}$$

The essence of the Fradkin method is to introduce an auxiliary, s-dependent field whose fluctuations reproduce the effects of noncommutivity of (∂_t,∇) and $\mathbf{v}$. Thus, the integrand of (7) is replaced by

$$U(s) = \Big(\exp\Big\{-\int_0^s ds' \Big(\Big[\partial_t - \nu\Big(\nabla - \frac{\mathbf{v}}{2\nu}\Big)^2 + \frac{\mathbf{v}^2}{4\nu} - \nu\mathbf{u}(s')\cdot\Big(\nabla - \frac{\mathbf{v}}{2\nu}\Big)\Big] - (\partial v)\Big)\Big\}\Big)_+, \tag{8}$$

where $\mathbf{u}(s')$ denotes a D-dimensional vector whose parametric dependence gives meaning to the s'-ordered exponential of (8); that is, in the expansion of $(\exp[-\int_0^s ds' A(s')])_+$, the noncommuting terms $A(s_1)\cdots A(s_n)$ are to be arranged in an ordered fashion, with those bearing the larger values of s_i standing to the left.[5] In terms of (8), (7) may be rewritten as

$$G[v] = \int_0^\infty ds\, U(s)|_{\mathbf{u}=0}. \tag{9}$$

The advantage of this procedure is that differential equations can now be written for $U(s)$, as a function of s and a functional of $\mathbf{u}$:

$$-\frac{\partial U}{\partial s} = \Big(\Big[\partial_t - \nu\Big(\nabla - \frac{\mathbf{v}}{2\nu}\Big)^2 + \frac{\mathbf{v}^2}{4\nu} - \nu\mathbf{u}(s)\cdot\Big(\nabla - \frac{\mathbf{v}}{2\nu}\Big)\Big] - (\partial v)\Big)U(s),$$

and

$$\frac{\delta U}{\delta \mathbf{u}(s)} = \nu\Big(\nabla - \frac{\mathbf{v}}{2\nu}\Big)U,$$

under the boundary condition $U(s=0) = 1$. The solution to this pair of equations can be written in the form

$$U(s) = \exp\Big[\frac{1}{\nu}\int_0^{s-\epsilon} ds' \frac{\delta^2}{\delta\mathbf{u}^2(s')}\Big]\cdot V(s), \tag{10}$$

where

$$V(s) = \Big(\exp\Big\{-\int_0^s ds' \times([\partial_t + \phi(\mathbf{x},x_0;s') - \nu\mathbf{u}(s')\cdot\nabla] - (\partial v))\Big\}\Big)_+,$$

with ϵ a small, positive parameter subsequently set equal to zero, and $\phi(\mathbf{x},x_0;s') = \mathbf{v}^2(\mathbf{x},x_0)/4\nu + \frac{1}{2}\mathbf{u}(s')\cdot\mathbf{v}(\mathbf{x},x_0)$. Strictly speaking, one cannot display $(\mathbf{x},x_0)$ dependence before taking matrix elements of $G[v]$, and at this stage it should be understood that we are calculating $\langle \mathbf{x},x_0|G[v]$, with the $V(s)$ of (10) replaced by $\langle \mathbf{x},x_0|V(s)$.

To find a representation for $V(s)$, one may set

$$V(s) = \exp\Big\{-\int_0^s ds' [\partial_t - \nu\mathbf{u}(s')\cdot\nabla]\Big\}\cdot W(s), \tag{11}$$

so that $W(s)$ satisfies the equation

$$-\frac{\partial W}{\partial s}=\exp\Big\{-\int_0^s ds'\,[\partial_t-\nu\mathbf{u}(s')\cdot\nabla]\Big\}$$
$$\times\{\phi(\mathbf{x},x_0;s)-[\partial v(\mathbf{x},x_0)]\}$$
$$\times\exp\Big\{+\int_0^s ds'\,[\partial_t-\nu\mathbf{u}(s')\cdot\nabla]\Big\}\cdot W,$$

or

$$-\frac{\partial W}{\partial s}=\Big[\phi\Big(\mathbf{x}-\nu\int_0^s ds'\,\mathbf{u}(s'),x_0+s;s\Big)-\Big(\partial v(\mathbf{x}-\nu\int_0^s ds'\,\mathbf{u}(s'),x_0+s)\Big)\Big]\cdot W. \quad (12)$$

Because of the particular sequence of translational operators written in (12), the $(\mathbf{x},x_0)$ dependence inside $W(s)$ is the same on both sides of (12); and hence (12) represents a differential equation that can be solved in terms of actual functions rather than formal operators.

Consider now the tensor quantity $\partial_b v_a(\mathbf{x},x_0)$, written in matrix notation as $Q_{ab}(\mathbf{x},x_0)$. Because $\mathbf{v}$ is assumed divergenceless, $\Sigma_a Q_{aa}=0$, and this $D\times D$ matrix is traceless. But any such matrix can be written in terms of the fundamental, or defining representation of SU(D),

$$Q_{ab}=\partial_b v_a=\sum_{i=1}^{D^2-1}(\lambda_i)_{ab}\psi_i(\mathbf{x},x_0), \quad (13)$$

where the ψ_i are a set of appropriate, complex, coefficient functions, forming a "vector" in the space of D^2-1 dimensions. For example, for $D=3$, the λ_i may be represented by the eight, traceless, Gell-Mann matrices.[6] Writing out all matrix indices, (12) takes the form

$$\frac{\partial W_{ac}}{\partial s}=-\Big[\delta_{ab}\phi\Big(\mathbf{x}-\nu\int_0^s ds'\,\mathbf{u}(s'),x_0+s;s\Big)-\sum_i(\lambda_i)_{ab}\psi_i\Big(\mathbf{x}-\nu\int_0^s ds'\,\mathbf{u}(s'),x_0+s\Big)\Big]W_{bc},$$

and, with the boundary condition $W_{ab}[s=0]=\delta_{ab}$, has the solution

$$W(s)=\exp\Big(-\int_0^s ds'\,\phi\Big(\mathbf{x}-\nu\int_0^{s'} ds''\mathbf{u}(s''),x_0+s';s'\Big)\Big)\cdot\Big(\exp\Big[\sum_i\lambda_i\int_0^s ds'\,\psi_i\Big(\mathbf{x}-\nu\int_0^{s'} ds''\,\mathbf{u}(s''),x_0+s'\Big)\Big]\Big)_+. \quad (14)$$

If the vortex stretching term was missing from the original Eq. (2), $(\omega\cdot\nabla)\mathbf{v}\to 0$, this would be equivalent to $\psi_i=0$, and $G[v]$ would be diagonal in SU(D) space. This is precisely the case for $D=2$, where the nonsinglet terms can always be "gauged away," and possibly for those situations in three dimensions where the vorticity source geometry may be sufficiently symmetric to enforce this "singlet" property. In general, however, the specification of vorticity in terms of a given velocity field is a nonabelian problem of the same order of difficulty as that of calculating a particle propagator in the presence of a specified field containing isotopic or color degrees of freedom. There is no known way of finding an exact solution to either problem.

For the moment, we designate the s' ordered bracket of (14) by the matrix symbol $\overline{U}[\nu\mathbf{u}]$, and postpone a discussion of its properties until the three-dimensional analysis of Sec. IV. All the effects of vortex stretching are contained in $\overline{U}$. For the two-dimensional examples of Sec. III, one may set $\overline{U}=1$, and study the properties of a singlet $G[v]$,

$$[\partial_t+(\mathbf{v}\cdot\nabla)-\nu\nabla^2]G[v]=1. \quad (5')$$

Grouping together the results of Eqs. (9)–(11) and (14), we have the representation

$$G[v]=\int_0^\infty ds\cdot\exp\Big[\frac{1}{\nu}\int_0^s ds'\,\frac{\delta^2}{\delta\mathbf{u}^2(s')}\Big]\cdot\exp\Big[-\int_0^s ds'\,\phi\Big(\mathbf{x}+\nu\int_{s'}^s ds''\,\mathbf{u}(s''),x_0-s+s';s'\Big)\Big]\cdot\overline{U}[\nu\mathbf{u}]\cdot\exp\Big\{-\int_0^s ds'\,[\partial_t-\nu\mathbf{u}(s')\cdot\nabla]\Big\}\Big|_{\mathbf{u}=0},$$

or

$$G(\mathbf{x},\mathbf{y};x_0,y_0|v)=\langle\mathbf{x},x_0|G[v]|\mathbf{y},y_0\rangle=\int_0^\infty ds\cdot\exp\frac{1}{\nu}\int_0^s ds'\,\frac{\delta^2}{\delta\mathbf{u}^2(s')}\cdot\exp\Big[-\int_0^s ds'\,\phi\Big(\mathbf{x}+\nu\int_{s'}^s ds''\,\mathbf{u}(s''),x_0-s+s';s'\Big)\Big]\cdot\overline{U}[\nu\mathbf{u}]\cdot\delta^D\Big(\mathbf{x}-\mathbf{y}+\nu\int_0^s ds'\,\mathbf{u}(s')\Big)\cdot\delta(x_0-y_0-s)\big|_{\mathbf{u}=0}, \quad (15)$$

where the $(\mathbf{x},x_0)$ arguments of $\overline{U}$ may be replaced by $(\mathbf{y},y_0)$. From the temporal δ function of (15), and the positive range of integration of the variable s, it is clear that this is a retarded Green's function, nonzero only for $x_0\geqslant y_0$.

Equation (15) is an example of the Fradkin representation, given in terms of the action of a functional differential operator,

$$\exp\Big[\frac{1}{\nu}\int_0^s ds'\,\frac{\delta^2}{\delta^2\mathbf{u}(s')}\Big],$$

à la Schwinger. For some purposes, however, it is more convenient to recast (15) into the form of functional integration, and effectively into a path integral, à la Feynman. Imagine the continuum range of integration broken up into a summation over a set of discrete points s_i, and for each s_i replace

$$\exp\Big[\frac{1}{\nu}\frac{\delta^2}{\delta\mathbf{u}^2(s_i)}\Big]$$

by its Gaussian equivalent. Then take the limit of arbitrarily dense s_i, to define the functional integral replacement

$$\exp\Big[\frac{1}{\nu}\int_0^s ds'\,\frac{\delta^2}{\delta\mathbf{u}^2(s')}\Big]=N(s)\cdot\int d[\chi(s)]\cdot\exp\Big[-\frac{\nu}{4}\int_0^s ds'\,\chi(s')^2+\int_0^s ds'\,\chi(s')\cdot\frac{\delta}{\delta\mathbf{u}(s')}\Big],$$

where

$$N^{-1}(s)=\int d[\chi(s)]\,\exp\Big[-\frac{\nu}{4}\int_0^s ds'\,\chi^2(s')\Big]. \quad (16)$$

With (16), any operation of form

$$\exp\Big[\frac{1}{\nu}\int_0^s ds'\,\frac{\delta^2}{\delta\mathbf{u}^2(s')}\Big]\cdot F\{\mathbf{u}\}|_{\mathbf{u}=0}$$

becomes $N(s)\cdot\int d[\chi]\,\exp[-(\nu/4)\int_0^s ds'\,\chi^2(s')]\cdot F\{\chi\}$, and

(15) may be rewritten as

$$G(\mathbf{x},\mathbf{y};x_0,y_0|v) = \theta(x_0 - y_0)\cdot N(s)\cdot \int d[\chi]\, \delta^D\Big(\mathbf{x} - \mathbf{y} + \nu \int_0^s ds' \mathbf{u}(s')\Big)\cdot \overline{U}[\nu\chi]$$

$$\cdot \exp\Big[-\frac{\nu}{4}\int_0^s ds'\, \chi^2(s') - \int_0^s ds'\, \phi\Big(\mathbf{x} + \nu\int_{s'}^s ds''\, \chi(s''), x_0 - s + s'; s'\Big)\Big]\Big|_{s = x_0 - y_0},$$

where ϕ is the same function as that of (10), with $\chi(s')$ replacing $\mathbf{u}(s')$. A final rescaling, $\chi(s') = \xi(s')/\nu$, puts this into the more useful, and indeed more compelling, form

$$G(\mathbf{x},\mathbf{y};x_0,y_0|v) = \theta(x_0 - y_0)\cdot N' \int d[\xi]\cdot \delta^D\Big(\mathbf{x} - \mathbf{y} + \int_0^s ds' \xi(s')\Big)\cdot \overline{U}[\xi]$$

$$\cdot \exp\Big[-\frac{1}{4\nu}\int_0^s ds' \Big(\xi(s') + \mathbf{v}(\mathbf{y} - \int_0^{s'} ds''\xi(s''), y_0 + s')\Big)^2\Big]\Big|_{s = x_0 - y_0}, \tag{17}$$

where

$$[N']^{-1} = \int d[\xi] \exp\Big[-\frac{1}{4\nu}\int_0^s ds'\, \xi(s')^2\Big],$$

and $\theta(x)$ denotes the unit positive step function, $\theta(x) = +1, x>0$, $\theta(x) = 0, x<0$. The $\overline{U}$ of (17) is now given by its form in (14), with the $(\mathbf{x}, x_0)$ arguments of ψ_i replaced by $(\mathbf{y},y_0)$. So far, this is an exact resolution of Eq. (5).

Even a cursory glance at (17) leaves no doubt about the next, appropriate step to be taken in the limit of small viscosity. As $\nu \to 0$, the only appreciable contribution to the functional integral will come from vectors $\xi(s')$ chosen to minimize the effective action

$$S[\xi] = \frac{1}{4\nu}\int_0^s ds' \Big[\xi(s') + \mathbf{v}\Big(\mathbf{y} - \int_0^{s'} ds''\, \xi(s''), y_0 + s'\Big)\Big]^2,$$

which vectors must then satisfy the Euler condition $\delta S/\delta\xi = 0$ at some $\xi^{(0)}(s_1)$, $s \geqslant s_1 \geqslant 0$, given by

$$0 = \int_0^s ds' \Big[\delta_{ab}\delta(s' - s_1) - \partial_a v_b\Big(\mathbf{y} - \int_0^{s'} ds''\, \xi(s''), y_0 + s'\Big)\cdot\theta(s' - s_1)\Big]\cdot\Big[\xi_b(s') + v_b\Big(\mathbf{y} - \int_0^{s'} ds''\xi(s''), y_0 + s_1\Big)\Big].$$

The appropriate solution to this equation is, clearly, that given by the vector $\xi^{(0)}(s')$ satisfying

$$\xi_b^{(0)}(s') + v_b\Big(\mathbf{y} - \int_0^{s'} ds''\, \xi^{(0)}(s''), y_0 + s'\Big) = 0, \tag{18}$$

since it is only the solution to (18) which can provide a nonvanishing contribution to the functional integral of (17), as $\nu \to 0$. Because of the $(\mathbf{y},y_0)$ dependence of the $\mathbf{v}$ of (18), $\xi^{(0)}$ is then an implicit function of these variables.

In the limit of small ν, we approximately evaluate (17) by expanding $\xi(s')$ about $\xi^{(0)}(s';\mathbf{y},y_0)$, and retaining only quadratic $(\xi - \xi^{(0)})$ dependence in S. The resulting functional integral is then Gaussian, and can be evaluated without difficulty; taking into account the normalization factor N', and dropping the superscript of $\xi^{(0)}$, one obtains

$$G(\mathbf{x},\mathbf{y};x_0,y_0|v)|_{\nu\to 0}$$
$$= \theta(x_0 - y_0)\cdot \exp(-(1/2)\,\mathrm{Tr}\ln[1 + Q])\cdot \overline{U}[\xi]$$
$$\cdot \delta^D\Big(\mathbf{x} - \mathbf{y} + \int_0^s ds'\, \xi(s';\mathbf{y},y_0)\Big)\Big|_{s = x_0 - y_0}, \tag{19}$$

where the determinantal factor is defined by

$$\mathrm{Tr}\ln[1 + Q] = \int_0^1 d\lambda \int_0^s ds_1 \sum_a \langle s_1,a|Q \cdot [1 + \lambda Q]^{-1}|s_1,a\rangle$$
$$= \int_0^1 d\lambda \int_0^s ds_1 \int_0^s ds_2 \sum_{a,b} \langle s_1,a|Q|s_2,b\rangle \cdot \langle s_2,b|[1 + \lambda Q]^{-1}|s_1,a\rangle, \tag{20}$$

with

$$\langle s_1,a|Q|s_2,b\rangle$$
$$= -\theta(s_1 - s_2)\,\partial_b v_a\Big(\mathbf{y} - \int_0^{s_1} ds'\, \xi(s';\mathbf{y},y_0), y_0 + s_1\Big)$$
$$- \theta(s_2 - s_1)\,\partial_a v_b\Big(\mathbf{y} - \int_0^{s_2} ds'\, \xi(s';\mathbf{y},y_0), y_0 + s_2\Big) + \int_0^s ds'$$
$$\cdot\theta(s' - s_1)\theta(s' - s_2)\ \ \partial_a v_c\Big(\mathbf{y} - \int_0^{s'} ds''\, \xi(s'';\mathbf{y},y_0), y_0 + s'\Big)$$
$$\cdot\partial_b v_c\Big(\mathbf{y} - \int_0^{s'} ds''\, \xi(s'';\mathbf{y},y_0), y_0 + s'\Big).$$

Corrections to (19) and (20), expressed in ascending powers of ν, can be generated in the standard way.

One very great simplification of these equations, the replacement of the determinantal factor of (19) by unity, can be seen from the following argument. Defining an operator R by its matrix elements

$$\langle a,s_1|R|b,s_2\rangle = -\theta(s_1 - s_2)\,\partial_b v_a\Big(\mathbf{y} - \int_0^{s_1} ds'\, \xi(s';\mathbf{y},y_0), y_0 + s_1\Big),$$

it is then possible to replace the operator $1 + Q$ by $(1 + R^T)(1 + R) = (1 + R)^T(1 + R)$, where the superscript T denotes "transposed" in both spatial ($\mathbf{y}$) and s variables. It follows that $\mathrm{Tr}\ln(1 + Q) = \mathrm{Tr}\ln(1 + R)^T + \mathrm{Tr}\ln(1 + R) = 2\,\mathrm{Tr}\ln(1 + R)$, and one may now examine the latter, simpler quantity,

$$2\sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l}\mathrm{Tr}[R^l].$$

But because R is a "retarded" operator [all matrix elements $\langle as_\alpha | R | bs_\beta \rangle$ proportional to factors $\theta(s_\alpha - s_\beta)$], the trace operation vanishes for $l \geqslant 2$; and because $\nabla \cdot \mathbf{v} = 0$, the trace operation also vanishes for $l = 1$. Hence this term vanishes, and $\exp[-\frac{1}{2}\,\mathrm{Tr}\ln(1+Q)]$ may be replaced by unity. However, the properties of the matrix elements of Q, or of R, are of considerable importance when one asks for small-ν corrections to (19).

The special properties of the extremum vector $\xi(s_1;\mathbf{y},y_0)$ are central to the solutions that follow. If, in terms of a specified $\mathbf{v}(\mathbf{x},x_0)$, the velocity vector ξ is to satisfy (18), it is easy to see that

$$\frac{d\xi}{ds'} = -\left[\frac{\partial \mathbf{v}}{\partial y_0} + (\mathbf{v}\cdot\nabla_y)\mathbf{v}\right], \qquad (21)$$

$\mathbf{v} = \mathbf{v}(\mathbf{y} - \int_0^{s} ds' \xi, y_0 + s')$, so that variation of the parameter s' corresponds to the full, nonlinear, "hydrodynamic" variation of $\mathbf{v}$ with respect to space and time. The rhs of (21) is not zero, as it would be for the motion of a simple shock,[7] but is given by whatever is forced upon ξ by the form of the specified velocity field $\mathbf{v}(x,t)$, *via* (18). We interpret $\xi(s';\mathbf{y},y_0)$ as the velocity vector of a fictitious particle, or passive marker, advected by $\mathbf{v}$ according to (18). If $\mathbf{v}$ is sufficiently smooth, ξ can always be developed in a power series in s',

$$\xi(s';\mathbf{y},y_0) \simeq -\mathbf{v}(\mathbf{y},y_0) + s'\cdot\left[\frac{\partial \mathbf{v}}{\partial y_0} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right]_{s'=0} + \cdots;$$

but, because $s = x_0 - y_0$ can be arbitrarily large, this will not in general be a practical way to construct ξ.

It will turn out to be useful to integrate (18) once and consider the position vector of this passive marker. Setting $\xi_\alpha(s';\mathbf{y},y_0) \equiv (dx_\alpha/ds')(s';\mathbf{y},y_0)$ and $\int_0^{s_1} ds'\,\xi(s';\mathbf{y},y_0) = \mathbf{x}(s_1;\mathbf{y},y_0) - \mathbf{x}(0;\mathbf{y},y_0) \equiv \Delta\mathbf{x}(s_1;\mathbf{y},y_0)$, we define the quantity $\mathbf{q}(s_1;\mathbf{y},y_0) \equiv \mathbf{y} - \Delta\mathbf{x}(s_1;\mathbf{y},y_0)$ as the marker's position vector, thereby replacing (18) by

$$\mathbf{q}(s_1;\mathbf{y},y_0) = \mathbf{y} + \int_0^{s_1} ds'\,\mathbf{v}(\mathbf{q}(s';\mathbf{y},y_0),y_0+s'). \qquad (22)$$

For a problem dealing with discrete, pointlike vortices, the $\mathbf{q}$ vectors turn out to represent the time-dependent position vectors of the vortices themselves.

With (3), (19), and (22), we are now in a position to calculate vorticity and its corresponding velocity in the $\nu \to 0$ limit:

$$\omega_\alpha(\mathbf{x},x_0) = \int d^D y \int dy_0\, G_{\gamma\delta}(\mathbf{x},\mathbf{y};x_0,y_0|v) g_\delta(\mathbf{y},y_0) \qquad (23a)$$

and

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_0 + \int d^D x\, G_0(\mathbf{r}-\mathbf{x})\nabla\times\omega(\mathbf{x},t), \qquad (23b)$$

where $G_0(Q)$ denotes the appropriate, D-dimensional Green's function, $G_0 = -\nabla^{-2}$. For simplicity and clarity, we henceforth use the three-dimensional form, $G_0(Q) = [4\pi|\mathbf{Q}|]^{-1}$, except when reference is made to $D = 2$. We shall assume the vorticity is of finite spatial extent, and freely drop surface terms, replacing (23b), for example, by

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_0 + \frac{1}{4\pi}\int d^3x \left[\nabla_r \cdot \frac{1}{|\mathbf{r}-\mathbf{x}|}\right]\times\omega(\mathbf{x},t). \qquad (23c)$$

Note that there is a zero contribution to the integrand of (23c) when x sweeps over the point $\mathbf{r}$, for, while apparently singular, that contribution vanishes by symmetry, as in the relation $\int d^3p\, \mathbf{p}(\mathbf{p}^2)^{-2} \exp[i\mathbf{p}\cdot(\mathbf{r}-\mathbf{x})]|_{|\mathbf{x}-\mathbf{r}|\to 0} \Rightarrow 0$.

Combining (23) and (18), we obtain

$$v_\alpha(\mathbf{r},t) = v_\alpha^{(0)} + \frac{\epsilon_{\alpha\beta\gamma}}{4\pi}\int d^3y \int_{-\infty}^{t} dy_0\, g_\delta(\mathbf{y},y_0) \times \frac{[\mathbf{r}-\mathbf{q}(s;\mathbf{y},y_0)]_\beta}{|\mathbf{r}-\mathbf{q}(s;\mathbf{y},y_0)|^3}\cdot \overline{U}_{\gamma\delta}[\mathbf{q}(s_1;\mathbf{y},y_0)], \qquad (24)$$

where $s = t - y_0$ and $\overline{U}_{\gamma\delta}[\mathbf{q}(s_1;\mathbf{y},y_0)]$ denotes $(\exp[\Sigma_i \lambda_i \int_0^s ds'\, \psi_i(\mathbf{q}(s';\mathbf{y},y_0),y_0+s')])_+$. Equations (22) and (24) represent our resolution of the inviscid NS problem.

With the aid of (24), one can rewrite (22) in the form

$$q_\alpha(s_1;\mathbf{y},y_0) = y_\alpha + v_\alpha^{(0)} s_1 + \frac{\epsilon_{\alpha\beta\gamma}}{4\pi}\int d^3y' \int_0^{s_1} ds' \int_{-\infty}^{y_0+s'} dy_0' \cdot \overline{U}_{\gamma\delta}\cdot g_\delta(\mathbf{y}',y_0') \cdot \frac{[q_\beta(s';\mathbf{y},y_0) - q_\beta(s';\mathbf{y}',y_0')]}{|\mathbf{q}(s';\mathbf{y},y_0) - \mathbf{q}(s';\mathbf{y}',y_0')|^3}, \qquad (25)$$

where the $\overline{U}$ of (25) denotes

$$(\exp[\Sigma_i \lambda_i \textstyle\int_0^{s'} ds''\, \psi_i(\mathbf{q}(s'';\mathbf{y}',y_0'),y_0'+s'')])_+.$$

It is perhaps not an easy equation to solve exactly, but there are, perhaps, reasonably simple vorticity source distributions where exact or reasonably simple approximations to (25) may be devised. Once one has solved, or approximated (25), knowledge of $\mathbf{q}(s;\mathbf{y},y_0)$ can then be used to construct the desired $\mathbf{v}(\mathbf{r},t)$. Equation (25) provides, in essence, a determining equation for the Cauchy Lagrangian-position coordinates, generalized to arbitrary source distributions $\mathbf{g}(\mathbf{y},y_0)$.

III. EXAMPLES

For simplicity we henceforth consider a "one-shot" vorticity source, turning on and off rapidly at $t \simeq 0$, and modeled by $\mathbf{g}(\mathbf{y},y_0) = \mathbf{g}(y)\cdot\delta(y_0)$; that is, at $t = 0$ a specified vorticity distribution is inserted into the fluid, and we watch the resulting fluid flow develop in time. Generalizations correspond to a continuous input of vorticity are easily written down for every example, but will not be considered in this paper. The examples of this section will deal with two-dimensional flows, for which we set $\overline{U} = 1$ and use the $D = 2$ form of $G_0 = -\nabla^{-2}$. In this way (24) and (25) become

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_0 + \frac{\theta(t)}{2\pi}\int d^2y\, \mathbf{g}(\mathbf{y})\times\frac{[\mathbf{r}-\mathbf{q}(t;\mathbf{y})]}{|\mathbf{r}-\mathbf{q}(t;\mathbf{y})|^2} \qquad (26)$$

and

$$\mathbf{q}(s_1;\mathbf{y}) = \mathbf{y} + \mathbf{v}_0 s_1 + \frac{1}{2\pi}\int_0^{s_1} ds' \int d^2y'\, \mathbf{g}(\mathbf{y}') \times \frac{[\mathbf{q}(s';\mathbf{y}) - \mathbf{q}(s';\mathbf{y}')]}{|\mathbf{q}(s';\mathbf{y}) - \mathbf{q}(s';\mathbf{y}')|^2}, \qquad (27)$$

where the $y_0 = 0$ coordinate of q has been suppressed. The radius vector is here given by $\mathbf{r} = \hat{\imath}x_1 + \hat{\jmath}x_2$, with the source $\mathbf{g}(y)$ pointing in the $\hat{k}$ direction; the vector $\mathbf{y}$ is understood to lie in the (x_1,x_2) plane. We discuss the solution of this pair of equations for the following three situations.

(A) One very thin vortex tube, of radius much less than any other length dimension, modeled by

$$\mathbf{g}(\mathbf{y}) = \mathbf{g}\delta(\mathbf{y}) = \mathbf{g}\delta(y)/2\pi y.$$

Here, both the y and y' variables of (26) and (27) are to be set equal to zero. Remembering the comment following (23c), which is also true in two dimensions, one has, directly, $\mathbf{q}(s_1;0) = \mathbf{v}_0 s_1$ as the solution of (27). There then follows

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_0 + \frac{\theta(t)}{2\pi}\mathbf{g}\times\frac{[\mathbf{r}-\mathbf{v}_0 t]}{|\mathbf{r}-\mathbf{v}_0 t|^2}, \tag{28a}$$

$$\boldsymbol{\omega}(\mathbf{r},t) = \theta(t)\mathbf{g}\delta(\mathbf{r}-\mathbf{v}_0 t). \tag{28b}$$

Equations (28) should be exact; that is, they should simultaneously satisfy the Euler equation $(\partial_t + (\mathbf{v}\cdot\nabla))\boldsymbol{\omega} = \mathbf{g}\delta(\mathbf{r})\delta(t)$, together with $\boldsymbol{\omega} = \nabla\times\mathbf{v}$. This is easily verified.

(B) A circular vortex sheet, modeled by a dense collection of very thin vortex tubes, all of equal strength, all pointing in the $\hat{k}$ direction, and arranged in a ring of radius r_0 in the (x_1, x_2) plane. In the limit of a continuous number of such thin tubes, we write $\mathbf{g}(\mathbf{y}) = (\mathbf{G}/r_0)\delta(y - r_0)$, and for simplicity set $\mathbf{v}_0 = 0$. In the (x_1, x_2) plane, we set $|\mathbf{y}| = |\mathbf{y}'| = r_0$, and $\mathbf{q}(s_1;\mathbf{y}) \to \mathbf{q}(s_1;\theta) = \hat{i}q_1(s_1;\theta) + \hat{j}q_2(s_1;\theta)$. Substituting these into the two-dimensional form of (27) yields the pair of equations

$$q_1(s;\theta) = r_0\cos\theta - G\int_0^s ds'\int_0^{2\pi}\frac{d\theta'}{2\pi} \times\frac{[q_2(s';\theta) - q_2(s';\theta')]}{|\mathbf{q}(s';\theta) - \mathbf{q}(s';\theta')|^2}, \tag{29a}$$

$$q_2(s;\theta) = r_0\sin\theta + G\int_0^s ds'\int_0^{2\pi}\frac{d\theta'}{2\pi} \times\frac{[q_1(s';\theta) - q_1(s';\theta')]}{|\mathbf{q}(s';\theta) - \mathbf{q}(s';\theta')|^2}, \tag{29b}$$

where $\mathbf{y} = \hat{i}r_0\cos\theta + \hat{j}r_0\sin\theta$.

Equations (29) can be solved with the aid of the ansatz

$$q_1 = \phi_a(s_1)\cos\theta - \phi_b(s)\sin\theta,$$
$$q_2 = \phi_a(s)\sin\theta + \phi_b(s)\cos\theta, \tag{30}$$

which, when substituted into (29), leads to the pair of equations

$$\phi_a(s) = r_0 - \frac{G}{2}\int_0^s ds'\frac{\phi_b(s')}{[\phi_a^2(s') + \phi_b^2(s')]}, \tag{31a}$$

$$\phi_b(s) = +\frac{G}{2}\int_0^s ds'\frac{\phi_a(s')}{[\phi_a^2(s') + \phi_b^2(s')]}. \tag{31b}$$

Equations (31) may be solved most simply by rewriting them as differential equations parametrized in the form

$$\phi_a(s) = \rho(s)\cos\psi(s),\ \ \phi_b(s) = \rho(s)\sin\psi(s).$$

One immediately finds that $\rho(s_1) = \text{const} = r_0$, while $\psi(s) = (G/2r_0^2)s$. Regrouping and substituting into (30), one has the solutions

$$q_1 = r_0\cos[\theta + (G/2r_0^2)s],\ \ q_2 = r_0\sin[\theta + (G/2r_0^2)s]. \tag{32}$$

Inserting (32) into (26), one finds

$$\mathbf{v}(\mathbf{r},t) = \frac{\theta(t)}{2\pi}\int_0^{2\pi} d\theta\,\mathbf{G}\times\frac{[\mathbf{r}-\mathbf{q}(t;\theta)]}{|\mathbf{r}-\mathbf{q}(t;\theta)|^2}. \tag{33}$$

The quadrature of (33) can be performed, and yields

$$\mathbf{v}(\mathbf{r},t) = \mathbf{G}\times\hat{r}\phi(r),\quad \boldsymbol{\omega}(\mathbf{r},t) = \theta(t)(\mathbf{G}/r_0)\delta(r - r_0), \tag{34}$$

where $\phi(r) = r^{-1}, r > r_0$, $\phi(r) = (2r_0)^{-1}, r = r_0$, and $\phi(r) = 0$, $r < r_0$. Again, (33) represents an exact solution to the inviscid Euler equation. It will be noted that the special angular symmetry of the vorticity source distribution effectively removes all the time dependence of the q_i of (32), resulting in the relatively simple form of (34), which indeed could have more easily been guessed from an inspection of the original Euler equation. However, in other situations where such symmetry is lacking, our method may turn out to be useful.

(C) Two thin vortices of arbitrary strength generate a soluble problem without the overwhelming symmetry of the previous example. We take $\mathbf{g}(\mathbf{y}) = \mathbf{g}_1\delta(\mathbf{y}-\mathbf{y}_1) + \mathbf{g}_2\delta(\mathbf{y}-\mathbf{y}_2)$, and write the pair of equations corresponding to (27),

$$\mathbf{q}_1(s) = \mathbf{y}_1 + \mathbf{v}_0 s + \frac{1}{2\pi}\int_0^s ds'\,\mathbf{g}_2\times\frac{[\mathbf{q}_1(s') - \mathbf{q}_2(s')]}{|\mathbf{q}_1(s') - \mathbf{q}_2(s')|^2}, \tag{35a}$$

$$\mathbf{q}_2(s) = \mathbf{y}_2 + \mathbf{v}_0 s + \frac{1}{2\pi}\int_0^s ds'\,\mathbf{g}_1\times\frac{[\mathbf{q}_2(s') - \mathbf{q}_1(s')]}{|\mathbf{q}_2(s') - \mathbf{q}_1(s')|^2}. \tag{35b}$$

Writing $\mathbf{q}(s) = \mathbf{q}_1(s) - \mathbf{q}_2(s)$, $\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2$, $\mathbf{G} = \mathbf{g}_1 + \mathbf{g}_2$, the difference of Eqs. (35) becomes

$$\mathbf{q}(s) = \mathbf{y} + \mathbf{G}\times\int_0^s ds'\frac{\mathbf{q}(s')}{|\mathbf{q}(s')|^2}, \tag{36}$$

which has a strong resemblance to the equations of example (B). Since $\mathbf{y}$ is a vector in the (x_1, x_2) plane, perpendicular to $\mathbf{G}$, we can use $\hat{y}$, $\hat{G}$ and $\hat{y}\times\hat{G}$ as three orthogonal directions, writing $\mathbf{q}(s) = \hat{y}F_1(s) + (\hat{y}\times\hat{G})F_2(s)$. Substitution into (36) then generates the pair of equations

$$F_1(s) = y - G\int_0^s ds'\frac{F_2(s')}{[F_1^2(s') + F_2^2(s')]}, \tag{37a}$$

$$F_2(s) = +G\int_0^s ds'\frac{F_1(s')}{[F_1^2(s') + F_2^2(s')]}, \tag{37b}$$

with solution [by comparison with Eqs. (31)]

$$F_1(s) = y\cos(Gs/y^2),\ \ F_2(s) = y\sin(Gs/y_2).$$

Now that $\mathbf{q}(s)$ is known, one returns to the original Eqs. (35) to construct, by simple quadrature, the individual $\mathbf{q}_1(s)$, $\mathbf{q}_2(s)$; for example,

$$\mathbf{q}_1(s) = \mathbf{y}_1 + \mathbf{v}_0 s + \frac{y}{G}\mathbf{g}_2 \times\Big\{\hat{y}\sin\Big(\frac{Gs}{y^2}\Big) + (\hat{y}x\hat{G})\Big[1 - \cos\Big(\frac{Gs}{y^2}\Big)\Big]\Big\},$$

and similarly for $\mathbf{q}_2(s)$. These $\mathbf{q}_l(t)$ are then inserted into the expression corresponding to (26) to obtain

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_0 + \frac{\theta(t)}{2\pi}\sum_{l=1}^{2}\mathbf{g}_l\times\frac{[\mathbf{r}-\mathbf{q}_l(t)]}{|\mathbf{r}-\mathbf{q}_l(t)|^2}. \tag{38}$$

From (38) one calculates

$$\boldsymbol{\omega}(\mathbf{r},t) = \theta(t)\sum_{l=1}^{2}\mathbf{g}_l\delta(\mathbf{r}-\mathbf{q}_l(t)), \tag{39}$$

showing that these vortices, inserted into the fluid at positions $\mathbf{y}_l$ at $t = 0$, subsequently move relative to the fixed coordinate system, or to each other, as specified by the time dependence of the $\mathbf{q}_l(t)$. Again, in this problem of very thin vortices, the "marker" coordinates have become those of the vortices; and, again, it is easy to show that (38) and (39) form an exact solution to the inviscid Euler equation. Of course, these solutions are well known; their usual differential equation, given in terms of complex position coordinates,[8] can be read off from the time derivative of Eqs. (35).

Generalizations to the case of N point vortices may be handled in the same manner. The complexity of the analysis, of course, increases rapidly with N, while for $N = 4$ there is evidence for a chaotic behavior of the solutions.[1]

IV. VORTEX STRETCHING

In this section we discuss some properties of the nonunitary matrix $\overline{U}$ relevant in three dimensions. For simplicity, the analysis continues to assume that $g(y,y_0) = g(y)\cdot\delta(y_0)$, using the source to insert an arbitrary vortex distribution into the fluid at $t = 0$. We suppress the $y_0 = 0$ coordinate of $\mathbf{q}(s; \mathbf{y}, y_0) = \mathbf{q}(s; \mathbf{y})$, and first inspect the $\mathbf{y}$ dependence of $\mathbf{q}$, as expressed by (22).

Under the variation $\mathbf{y} \to \mathbf{y} + \delta\mathbf{y}$,

$$\mathbf{q}(s;\mathbf{y} + \delta\mathbf{y}) = \mathbf{y} + \delta\mathbf{y} + \int_0^s ds'\, \mathbf{v}(\mathbf{q}(s';\mathbf{y} + \delta\mathbf{y}),s'),$$

or to first order in $\delta\mathbf{y}$,

$$(\delta\mathbf{y}\cdot\nabla)q_c(s;\mathbf{y}) = \delta y_c + \int_0^s ds'\, [(\delta\mathbf{y}\cdot\nabla)q_a(s';\mathbf{y})] \cdot[\partial_a v_c(\mathbf{q}(s';\mathbf{y}),s')].$$

If $\delta\mathbf{y}$ has a nonzero component only in the b direction, this becomes

$$\partial_b q_c(s;\mathbf{y}) = \delta_{cb} + \int_0^s ds'\, \partial_b q_a(s';\mathbf{y})\cdot\partial_a v_c(\mathbf{q}(s';\mathbf{y}),s'). \quad (40)$$

But (40) is just an expression of the integral equation whose solution is the $\overline{U}$ of (25), since the quantity $Q_{ca}(s) \equiv \partial_a v_c(\mathbf{q}(s;\mathbf{y}),s) = \Sigma_i (\lambda_i)_{ca}\psi_i(\mathbf{q}(s;\mathbf{y}),s)$ is precisely the interaction term of the differential equation built from (40),

$$\frac{\partial \overline{U}_{cb}}{\partial s} = Q_{ca}(s)\cdot\overline{U}_{ab}(s), \quad (41)$$

where we have written $\overline{U}_{cb}(s) \equiv \partial_b q_c(s;\mathbf{y})$, in anticipation of this result. Thus, the exact solution to (41) is

$$\overline{U}(s) = \left(\exp\left[\int_0^s ds'\, Q(s')\right]\right)_+, \quad (42)$$

which is precisely the ($y_0 = 0$) quantity $\overline{U}$ of (25).

Knowing that the vortex stretching term $\overline{U}_{cb}$ is nothing other than the spatial gradient of the vortex source, or marker, coordinate $\partial_b q_c(s;\mathbf{y})$ is interesting[3] and may even turn out to be of some practical use in finding an approximate solution to (25). For the qualitative purpose of this paper, however, we will examine the behavior of $\overline{U}$ in terms of its SU(3) coordinates.[9] The first task is to define the ψ_i, which follow from the definition of Q and the properties of the Hermitian λ_i.

For convenience, these properties—taken from Ref. 4—are grouped together in the Appendix, and will be used as needed.

From the definition, $\partial_b v_a = (\lambda_i)_{ab}\psi_i$, application of the trace property $\mathrm{Tr}[\lambda_i\lambda_j] = 2\delta_{ij}$ leads to $\psi_i = \frac{1}{2}\Sigma_{a,b}(\lambda_i)_{ba}\,\partial_b v_a$. From the explicit forms of the Appendix, one then constructs the components

$$\begin{aligned}
&\psi_1 = \tfrac{1}{2}[\partial_1 v_2 + \partial_2 v_1], \quad \psi_2 = \tfrac{1}{2}i[-\partial_1 v_2 + \partial_2 v_1],\\
&\psi_3 = \tfrac{1}{2}[\partial_1 v_1 + \partial_2 v_2], \quad \psi_4 = \tfrac{1}{2}[\partial_1 v_3 + \partial_3 v_1],\\
&\psi_5 = \tfrac{1}{2}i[-\partial_1 v_3 + \partial_3 v_1], \quad \psi_6 = \tfrac{1}{2}[\partial_2 v_3 + \partial_3 v_2],\\
&\psi_7 = \tfrac{1}{2}i[-\partial_2 v_3 + \partial_3 v_2],\\
&\psi_8 = (1/\sqrt{3})[\partial_1 v_1 + \partial_2 v_2 - 2\,\partial_3 v_3].
\end{aligned} \quad (43)$$

Since each Q_{ab} must be real, the three components ψ_2, ψ_5, ψ_7 are imaginary because the corresponding $\lambda_2, \lambda_5, \lambda_7$ have purely imaginary components. It is interesting to note that these three ψ_i are just proportional to the three components of vorticity, ω_a; while the remaining ψ_i are given by real symmetric velocity gradients. It will be useful to divide this collection of real and imaginary terms into two sets: $\lambda_i\psi_i \equiv i\lambda_a\omega_a + \lambda_\alpha\phi_\alpha$, where the λ_a run over the imaginary matrices, $a = (2, 5, 7)$, and the λ_α run over the real matrices, $\alpha = (1, 3, 4, 6, 8)$, with the ϕ_α corresponding to the real, symmetric, velocity gradient components of ψ. It should be emphasized that all of these $\lambda_a, \lambda_\alpha$ are Hermitian.

Equation (41) can then be rewritten in the form

$$\frac{\partial \overline{U}}{\partial s} = (i\lambda\cdot\omega + \lambda\cdot\phi)\overline{U}, \quad \overline{U}(s = 0) = 1, \quad (44)$$

and the nonunitary nature of the vortex stretching becomes clear upon writing the Hermitian conjugate of (44), and adding the two equations to obtain

$$\frac{\partial}{\partial s}(\overline{U}^\dagger\overline{U}) = 2\overline{U}^\dagger\,\lambda\cdot\phi\overline{U}, \quad (45)$$

showing that it is the symmetric components ϕ_α which govern the increase of $\overline{U}^\dagger\overline{U}$. The s dependence of ϕ, as of ω, is contained in the space-time $(\mathbf{q}(s;\mathbf{y}),s)$ arguments of these functions.

In order to construct solutions to (44), it is useful to extract the manifestly Hermitian part of $\overline{U}$ by setting $\overline{U}(s) = \overline{V}(s)\cdot\overline{W}(s)$, with $\overline{V}(s) = (\exp[i\int_0^s ds'\,\lambda\cdot\omega(s')])_+$, where the ordering symbol refers to the s' variable, as in (8). The matrix $\overline{V}(s)$ is then unitary, $\overline{V}^\dagger = \overline{V}^{-1}$, and all the nonunitary behavior of $\overline{U}$ can be transferred to $\overline{W}$, which satisfies

$$\frac{\partial \overline{W}}{\partial s} = \lambda\cdot\Phi(s)\overline{W}(s), \quad \overline{W}(0) = 1, \quad (46)$$

where $\lambda\cdot\Phi = \overline{V}^\dagger(s)\lambda\cdot\phi(s)\overline{V}(s)$. The real components, $\Phi_i(s)$ are given by

$$\Phi_i = \tfrac{1}{2}\,\mathrm{Tr}[\lambda_i\overline{V}^\dagger\lambda\cdot\phi\overline{V}], \quad (47)$$

and it is also true that the magnitudes of Φ_i and ϕ_α are equal,

$$\Phi^2 \equiv \sum_i \Phi_i^2 = \sum_\alpha \phi_\alpha^2 \equiv \phi^2. \quad (48)$$

From (46), it follows that

$$\frac{\partial}{\partial s}(\overline{W}^\dagger\overline{W}) = 2\overline{W}^\dagger(\overline{V}^\dagger\lambda\cdot\phi\overline{V})\overline{W},$$

which is consistent with (45) and the unitarity of $\overline{V}$, $\overline{W}^\dagger\overline{W} = \overline{U}^\dagger\overline{U}$.

The way in which these properties shall be used is as follows. We begin with the "vectors" ϕ_α and ω_a, and assume that all velocity gradients are increasing in magnitude with time ($s \equiv t$ when $y_0 = 0$), while their "angular" variations are relatively slowly varying. That is, we take $\boldsymbol{\omega} = \hat{\omega}\omega$, $\hat{\omega}^2 \equiv 1$, and assume that

$$\left| \frac{d\hat{\omega}}{dt} \right| \ll \omega, \quad \int_0^t dt' \, \omega(t') \gg 1. \tag{49}$$

A corresponding relation for the angular behavior of the symmetric components $\boldsymbol{\phi} = \hat{\phi}\phi$ will not be necessary, although we continue to assume that the time integral of the magnitude of ϕ increases in a reasonably steady way, $\int^t dt' \, \phi(t') \gg 1$, while $\hat{\phi}$ does not change too rapidly in time. These assumptions will be justified, in a crude way, *a posteriori*.

Under these conditions we are interested in finding an approximate solution[9] for $\overline{W}(t)$, and begin by considering an explicit representation for $\overline{V}(t)$, which satisfies the differential equation

$$\frac{\partial \overline{V}}{\partial t} = i(\boldsymbol{\lambda}\cdot\boldsymbol{\omega}(t))\overline{V}(t), \quad \overline{V}(0) = 1. \tag{50}$$

Because $\overline{V}(t)$ is unitary, it may be written in the form $\overline{V}(t) = \exp[iF_0(t) + i\boldsymbol{\lambda}\cdot\mathbf{F}(t)]$, where $\mathbf{F}$ is, in general, a vector with eight real components. The initial condition $\overline{V}(0) = 1$ then becomes $F_0(0) = F_i(0) = 0$. Substituting this form of $\overline{V}(t)$ into (50), with the aid of the general formula

$$\frac{d}{dt} e^{\Gamma(t)} = \int_0^1 d\mu \, e^{\mu\Gamma(t)} \frac{d\Gamma}{dt} e^{(1-\mu)\Gamma(t)}, \tag{51}$$

useful when $[\Gamma, d\Gamma/dt] \neq 0$, one immediately learns that $F_0(t) \equiv 0$, as one builds a differential equation for $F(t)$,

$$\boldsymbol{\lambda}\cdot\boldsymbol{\omega}(t) = \int_0^1 d\mu \, e^{i\mu\boldsymbol{\lambda}\cdot\mathbf{F}(t)} \cdot \left(\boldsymbol{\lambda}\cdot \frac{d\mathbf{F}}{dt} \right) \cdot e^{-i\mu\boldsymbol{\lambda}\cdot\mathbf{F}(t)},$$

or

$$\omega_i(t) = \frac{1}{2} \int_0^1 d\mu \, \mathrm{Tr}\left[\lambda_i e^{i\mu\boldsymbol{\lambda}\cdot\mathbf{F}} \left(\boldsymbol{\lambda}\cdot \frac{d\mathbf{F}}{dt} \right) e^{-i\mu\boldsymbol{\lambda}\cdot\mathbf{F}} \right]. \tag{52}$$

It follows from (52), by multiplication by $\Sigma_i \hat{F}_i$, and the trace properties of the λ_j, that

$$\hat{F}\cdot\boldsymbol{\omega} = \hat{F}\cdot \frac{d\mathbf{F}}{dt} = \frac{dF}{dt},$$

where $F = \sqrt{\mathbf{F}^2} = (\Sigma_i F_i^2)^{1/2}$. Further, if $\mathbf{F}$ is written as $\hat{F}F$, (52) can be rewritten as

$$\omega_i(t) = \hat{F}_i \frac{dF}{dt} + \frac{F}{2} \int_0^1 \mu \, \mathrm{Tr}\left[\lambda_i e^{i\mu\boldsymbol{\lambda}\cdot\mathbf{F}} \left(\boldsymbol{\lambda}\cdot \frac{d\hat{F}}{dt} \right) e^{-i\mu\boldsymbol{\lambda}\cdot\mathbf{F}} \right]. \tag{53}$$

An alternate representation for (53) is obtained by expanding the exponentials of the second term, using the basic commutation relation of this Lie algebra, $[\lambda_i, \lambda_j] = 2if_{ijk}\lambda_k$, where the f_{ijk} structure constants are real, completely antisymmetric numbers as written in the Appendix. Introducing the adjoint representation, Hermitian matrices, $(A^i)_{jk} \equiv if_{ijk}$, (59) may be rewritten as

$$\omega_i(t) = \hat{F}_i \frac{dF}{dt} + F \int_0^1 d\mu \, \frac{d\hat{F}_1}{dt} (e^{2i\mu\mathbf{A}\cdot\mathbf{F}})_{ji},$$

or

$$\omega_i(t) = \hat{F}_i \frac{dF}{dt} + \frac{i}{2} \left(\frac{d\hat{F}_j}{dt} \right) [(\mathbf{A}\cdot\hat{F})^{-1}(1 - e^{2i\mathbf{A}\cdot\hat{F}F})]_{ji}. \tag{54}$$

Assuming that $(A\cdot F)^{-1}$ exists, that is, $\det(A\cdot F) \neq 0$, Eq. (54) is exact. It may be used to infer the form of $\mathbf{F}$ in the special circumstance that $F \gg 1$ and $|d\hat{F}/dt| \ll |dF/dt|$ or ω, for the second rhs term of (54) may be discarded in comparison to the first; because all the components of $\mathbf{F}$, or $\hat{F}$, are real numbers, and the $\mathbf{A}$ are Hermitian, the exponential of (54) simply oscillates rapidly as F increases. In this limit, (54) reduces to

$$\omega_i \simeq \hat{F}_i \frac{dF}{dt}, \tag{55}$$

which in comparison with (52), yields the approximate "strong-coupling" solution,

$$\hat{F}_i(t) = \hat{\omega}_i, \quad F(t) = \int_0^t dt' \, \omega(t'). \tag{56}$$

The conditions for the validity of this approximate solution,

$$\overline{V}(t) \simeq \exp\left[i\boldsymbol{\lambda}\cdot\hat{\omega}(t) \int_0^t dt' \, \omega(t') \right], \tag{57}$$

are those of (49).

One can now see in a clear way, from (47) and (48), the effect of such a unitary $\overline{V}$ under the supposed conditions (49). Writing the components Φ_i as $\Phi\hat{\Phi}_i$, where $\Phi = (\Sigma_i \Phi_i^2)^{1/2}$, it follows that the magnitude $\Phi(t) = \phi(t)$ is unchanged by the $\overline{V}$ unitary transformation, while the unit vector $\hat{\Phi}(t)$ is given a rapidly oscillating time dependence, with a frequency proportional to $(1/t)\oint_0^t dt' \, \omega(t')$. That is, if one writes an alternate expression for (57), in terms of the vector $\mathbf{Q} = \hat{\omega}\int_0^t dt' \, \omega(t')$,

$$\overline{V}(t) = \exp[i\mathbf{Q}\cdot\boldsymbol{\lambda}] \equiv f_0 + i\sum_a \lambda_a f_a + \sum_\alpha \lambda_\alpha g_\alpha,$$

where $f_0 = \frac{1}{3}\,\mathrm{Tr}\,[\exp(i\mathbf{Q}\cdot\boldsymbol{\lambda})]$, $f_a = -\frac{1}{2}i\,\mathrm{Tr}[\lambda_a \exp(i\mathbf{Q}\cdot\boldsymbol{\lambda})]$, $g_\alpha = \frac{1}{2}\,\mathrm{Tr}[\lambda_\alpha \exp(i\mathbf{Q}\cdot\boldsymbol{\lambda})]$. The quantity f_0 can be calculated directly from the three eigenvalues of the matrix $\mathbf{Q}\cdot\boldsymbol{\lambda}$, which satisfy the relations $\Sigma_{n=1}^3 \xi_n = 0$, $\Sigma_{n=1}^3 \xi_n^2 = 2\mathbf{Q}^2$, and $\Sigma_{n=1}^3 \xi_n^3 = -3D \equiv 2\Sigma_{ijk} d_{ijk} Q_i Q_j Q_k$. For the above choice of the Q_a, it can be seen that $D = 0$, following from the explicit d_{abc} of the Appendix, so that $\xi_1 = -\xi_3 = +Q \equiv j\sqrt{\mathbf{Q}^2}$, $\xi_2 = 0$, generate f_0 and $f_a = -\frac{3}{2}(\partial/\partial Q_a) f_0$. But the functions g_α are not directly calculable in this way, although they may be inferred in an indirect way from the requirement of unitarity.

A more elegant way of calculating both f_a and g_α at the same time is to write $\exp[i\mathbf{Q}\cdot\boldsymbol{\lambda}] = f_0 + i\Sigma_{i=1}^8 \lambda_i f_i$ and imagine that $\mathbf{Q}$ has projections in all eight directions, although its extension in the α directions is very small. Then $|D| \ll Q^2$, and the cubic equation for the ξ_n is easily approximated to yield

$$\xi_1 \simeq Q - D/2Q^2, \quad \xi_2 \simeq +D/Q^2, \quad \xi_3 \simeq -Q - D^2/2Q^2,$$

so that $f_0(Q) \simeq \frac{1}{3}[1 + 2\cos Q] + \frac{1}{3}i(D/Q^2)[1 - \cos Q]$, and the $f_i = -\frac{3}{2}(\partial/\partial Q_i) f_0$ can be evaluated in the $D \to 0$ limit as $f_a = \hat{Q}_a \sin Q$, $f_\alpha = iv_\alpha(1 - \cos Q) \equiv -ig_\alpha$, where $v_\alpha \equiv \Sigma_{ab\alpha} d_{ab\alpha} \hat{Q}_a \hat{Q}_b$. Note that all the matrix elements of $\overline{V}$

are real,

$$\overline{V}(t) = \tfrac{1}{3}[1 + 2\cos Q] + i\boldsymbol{\lambda}\cdot\hat{\omega}\sin Q - \sum_\alpha \lambda_\alpha v_\alpha (1-\cos Q). \quad (58)$$

Using the property $v^2 \equiv \Sigma_\alpha v_\alpha^2 = \frac{1}{3}$, also found via the Appendix, it is easy to verify unitarity; e.g., $1 = f_0^2 + \frac{2}{3}\Sigma_a f_a^2 + \frac{2}{3}\Sigma_\alpha g_\alpha^2$.

The effect of such a $\overline{V}$ unitary transformation, as in (47), is to make the unit vectors $\hat{\Phi}_i$ rotate with frequency proportional to $(1/t)\int_0^t dt'\,\omega(t')$, here assumed to be a reasonably large number, while leaving the magnitude unchanged, $\Phi = \phi$. One then tries to solve (46) under these conditions, and in a more explicit and useful form than that given by the formal solution,

$$\overline{W}(t) = (\exp[\textstyle\int_0^t dt'\,\boldsymbol{\lambda}\cdot\boldsymbol{\Phi}(t')])_+.$$

If one adopts the ansatz[10] $\overline{W}(t) = \exp[G_0(t) + \boldsymbol{\lambda}\cdot\mathbf{G}(t)]$, and forms the differential equation for $(G_0,\mathbf{G})$ using (51), one immediately obtains $G_0 \equiv 0$, and in a manner analogous to (54),

$$\Phi_i = \hat{G}_i\frac{dG}{dt} + G\int_0^1 d\mu\,\frac{d\hat{G}_j}{dt}\,(e^{2\mu\mathbf{A}\cdot\hat{G}G})_{ji}. \quad (59)$$

For large G, which we also assume and which turns out to be a more stringent assumption than $\int^t dt'\,\phi(t') \gg 1$, the exponent of the second rhs term of (59) cannot be treated as oscillatory, as in the analysis of (54), for the eigenvalues of $\mathbf{A}\cdot\hat{G}$ are real numbers, of both signs. It is still true, however, without approximation that

$$\frac{dG}{dt} = \hat{G}\cdot\boldsymbol{\Phi} = \xi(t)\phi(t),$$

or

$$G(t) = \int_0^t dt'\,\xi(t')\phi(t'), \quad (60)$$

where $\xi(t) = \hat{G}(t)\cdot\hat{\Phi}(t)$. Rewriting (59) in the form

$$\hat{\Phi}_i\phi - \hat{G}_i(\hat{G}\cdot\hat{\Phi})\phi = G\int_0^1 d\mu\,\frac{d\hat{G}_j}{dt}\,(e^{2\mu\mathbf{A}\cdot\hat{G}G})_{ji}, \quad (61)$$

and, multiplying both sides of (61) by $(\mathbf{A}\cdot\hat{G})_{il}$, one forms

$$\frac{1}{2}\frac{d\hat{G}_j}{dt}(e^{2\mathbf{A}\cdot\hat{G}G} - 1)_{jl} = \phi\Phi_i(\mathbf{A}\cdot\hat{G})_{il}, \quad (62)$$

with the quantity proportional to $\hat{G}_i(\mathbf{A}\cdot\hat{G})_i$ vanishing by symmetry. Just as one performed the analysis expressing the approximate form of $\overline{V}$ in terms of the eigenvalues of $\boldsymbol{\lambda}\cdot\mathbf{Q}$, as in (58), so the eigenvalues of $(\mathbf{A}\cdot\hat{G})$ can be invoked to rewrite (62). For large G, only the largest positive eigenvalue, $\eta_{\max}$, will be important, $e^{2(\mathbf{A}\cdot\hat{G})G} \sim e^{2G\eta_{\max}}\cdot O(1)$, and there follows from (62) the qualitative expression

$$\frac{d\hat{G}}{dt} \sim e^{-2G\eta_{\max}}\cdot\phi\hat{\Phi}(\mathbf{A}\cdot\hat{G}),$$

showing that, for large G, $d\hat{G}/dt$ is damped exponentially, i.e., after a certain time has been reached for which $G \gg 1$, $\hat{G}(t)$ is essentially constant. The unit vector $\hat{\Phi}(t)$, however, continues to oscillate wildly; and the result is that $\xi(t) = \hat{G}(t)\cdot\hat{\Phi}(t)$ oscillates to zero for sufficiently large t. For small t, $\hat{G}$ and $\hat{\Phi}$ are in phase, leading to $\xi(t) \sim 1$; but as time increases, they fall out of phase, and then in phase again, etc. Because G is supposed large and positive, $\phi(t)\xi(t)$ is more often positive than negative, of qualitative form given in Fig. 1. If this picture is correct, then $G(t)$ does not grow smoothly, but only in spurts, when $\xi(t)$ is reasonably positive. Since the onset of turbulence will be associated with $G \gg 1$, the behavior of $\xi(t)$ provides a possible mechanism for the fact of intermittency.[11] A detailed analysis depends upon a systematic resolution of (59). Finally, since $\overline{W} = \exp(\boldsymbol{\lambda}\cdot\hat{G}G)$, an eigenvalue analysis of $\boldsymbol{\lambda}\cdot\hat{G}$ will generate $\overline{W} \sim e^{G(t)}\cdot O(1)$, which is the only part of this discussion to be used below. In particular, for the nonunitary matrix $\overline{W}$,

$$\overline{W}^\dagger\overline{W} \sim e^{2G(t)}\cdot O(1). \quad (63)$$

To see how these SU(3) forms can be useful, we now insert them into a crude, dimensional model which examines the growth of enstrophy, $\Omega(t) = \frac{1}{2}\int d^3x\,\omega^2(\mathbf{x},t)$. With (23a) and (19), Ω can be put into the form

$$\Omega = \tfrac{1}{2}\int d^3y \int d^3y' \times\delta(\mathbf{q}(t;\mathbf{y}) - \mathbf{q}(t;\mathbf{y}'))g^\dagger(\mathbf{y}') \cdot\overline{W}^\dagger(\mathbf{q}(t;\mathbf{y}'),t)\overline{W}(\mathbf{q}(t;\mathbf{y}),t)g(\mathbf{y}),$$

using an obvious matrix notation. It is simplest to change variables from $(\mathbf{y},\mathbf{y}')$ to $(\mathbf{q},\mathbf{q}')$, observing that the determinant of each transformation is unity, $d^3q = \det(\partial q)\cdot d^3y$; and with (42), $\det(\partial q) = \exp(\mathrm{tr}\ln\overline{U}) \Rightarrow 1$. Then, (64) can be written as

$$\Omega = \tfrac{1}{2}\int d^3q\, g^\dagger(y(q))\cdot\overline{W}^\dagger\,\overline{W}\cdot g(y(q)),$$

or with (63)

$$\Omega \sim \int d^3q\cdot g\cdot g^\dagger\cdot e^{2G(t)}\cdot O(1)\cdot g. \quad (64)$$

We now invoke a crude, dimensional argument to determine the possible significance of the above assumptions, by transferring them to the behavior of a time-dependent length scale, $l(t)$. If only a single length scale is used, we assume we are studying the growth of "isotropic" turbulence, as represented by the growth of enstrophy.[12] What is fixed in the fluid is assumed to be its energy/density, $\kappa = E/\rho = \frac{1}{2}\int d^3x\,\mathbf{v}^2$, and we write the dimensional relation $\kappa \sim l^3v^2$, or $v \sim \kappa^{1/2}\cdot l^{-3/2}$. Then, $\Omega = \frac{1}{2}\int d^3x\,\omega^2 \sim l^3(v/l)^2 \sim \kappa/l^2$. Comparing with the rhs of (64), the dimensions of $\int d^3q\,g^\dagger g$ are also $\sim L^{-2}$, but these coordinates must refer to the initial configuration of the velocity/vortex fields, since the latter are introduced by the source g at $t = 0$. Hence we write $\int d^3q\,g^\dagger g \sim \kappa/l_0^2$, where l_0 refers to a typical length scale at a

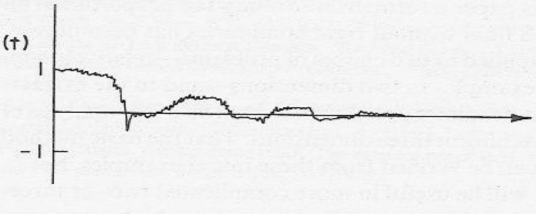

FIG. 1. Expected, qualitative form of $\xi(t)$, expressing the feature of intermittency.

pre-growth-of-turbulence time t_0, which for simplicity is set equal to zero, $t_0 = 0$. Finally, we estimate $G(t) = \int_0^t dt'\, \xi(t')\phi(t')$, using $\phi(t) \sim (v/l) \sim \kappa^{1/2}/l^{5/2}(t)$. In this way, a crude model of $l(t)$ can be extracted from (64),

$$(\kappa/l^2(t)) \sim (\kappa/l_0^2)\cdot\exp\left[2\kappa^{1/2}\int_0^t dt'\, \xi(t') l^{-5/2}(t')\right],$$

or

$$l(t) \sim l_0 \exp\left[-\kappa^{1/2}\int_0^t dt'\, \xi(t')\cdot l^{-5/2}(t')\right]. \quad (65)$$

The decrease of $l(t)$ as t increases presumably corresponds to the transfer of the original input energy into vortices of smaller and smaller spatial scale. To solve (65), it is only necessary to differentiate once and form the differential equation

$$\frac{dl}{dt} \sim -\kappa^{1/2}\xi(t)\cdot l^{-3/2}(t),$$

which can be integrated immediately,

$$l(t) = \left[l_0^{5/2} - \tfrac{5}{2}\kappa^{1/2}\int_0^t dt'\xi(t')\right]^{2/5},$$

generating

$$\Omega(t) \sim \left[l_0^{5/2} - \tfrac{5}{2}\kappa^{1/2}\int_0^t dt'\, \xi(t')\right]^{-4/5}. \quad (66)$$

It is clear that the type of growth of Ω, as t increases, now depends upon the falloff of $\xi(t)$, which we crudely model according to the following possibilities:

(i) If $\xi(t) \sim +t^{-(1-p)}$, $0<p<1$, then Ω has a finite-time singularity, $\Omega(t) \sim (t^* - t)^{-4/5}$.

(ii) if $\xi(t) \sim +t^{-(1+p)}$, $p>0$, Ω may or may not have a finite-time singularity, depending on the specific constants involved. The singularity[13] of possibility (i) is less severe than that previously found in the quasinormal approximation[14] $\Omega \sim (t^* - t)^{-2}$, but it may well be that possibility (ii) is more accurate.

If it is initially assumed that there are two relevant length scales, rather than one, e.g., $L(t)$ corresponding to the length of typical vortex, and $l(t)$ corresponding to its radial dimension, an analysis similar to that leading to (66) yields

$$l^2(t) \simeq l_0^2 - 2\kappa^{1/2}\int_{t_0}^t dt'\, \xi(t')L^{-1/2}(t'),$$

and possibilities similar to (i) and (ii) can be developed. In any case, nonzero viscosity will smooth out an inviscid finite-time singularity.

V. SUMMARY

In this paper a formalism to study the properties of an inviscid NS fluid without rigid boundaries has been developed and applied to two classes of problems—relatively simple, exact examples in two dimensions—and to the extraction of a crude, dimensional model from an SU(3) analysis of vortex stretching in three dimensions. That the basic method is correct can be verified from these initial examples, but whether it will be useful in more complicated two- or three-dimensional problems is a different matter, which remains to be seen.

To the best of the author's knowledge the application of Fradkin's generic Green's function representation, and the use of an SU(3) description for the ensuing vortex stretching, have not previously appeared in the (voluminous!) NS literature; and it is hoped that these techniques will provide tools for calculations less crude than those presented in Sec. IV. One interesting feature of the SU(3) analysis of strong vortex stretching has been the automatic appearance of a certain degree of intermittency, characterized by the function $\xi(t)$ of (60), which may turn out to be a useful way of describing the properties of experimental, intermittent turbulence. This tentative identification of $\xi(t)$ with intermittency is here only suggested, rather than claimed; but it is, perhaps, a suggestion which may turn out to be at least partially true.

The formalism itself suggests various other calculational attempts, such as the approximate solution of (25) when vortex stretching is represented by $\partial_a q_b$, or the behavior of these estimates when vortices are continually fed into the fluid, with the aid of a source $g(\mathbf{y},y_0)$ more general than the $\mathbf{G}(\mathbf{y})\cdot\delta(y_0)$ used above. Perhaps the most interesting modification would be the inclusion of viscosity corrections to all the calculations of this paper. A somewhat different resolution of Fradkin's original representation, not tied to the extremum calculation of Sec. II and valid for $\nu \neq 0$, will be presented separately.

It should be mentioned that hardly any methods exist to treat the nonabelian, vortex-stretching problem, when perturbation in $(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v}$ is improper, other than the strong-coupling approach of Sec. IV. Other possibilities are to treat the dimension D as very large, and search for simplifications in the large-D limit, in analogy with current work on the large-N limit of certain[15] quantum field theories; or for fixed D to replace the matrices λ_i, of the defining representation of SU(D), by semiclassical coordinates corresponding to higher-dimensional representation of SU(D). For example, for SU(3), the hypercharge and isospin quantum numbers present in this formalism could be treated as if they were continuous coordinates, following an approximation technique long known in nuclear and particle physics. But these are just stop-gap measures, which really do not get to the heart of the problem of how to find a useful, nonformal representation for the ordered bracket of (14).

The appearance of such SU(D) coordinates suggests that there may be an underlying SU(D) symmetry of the basic velocity/vorticity NS equations. Clearly, an analogy exists between certain aspects of hydrodynamics and nonabelian field theory; perhaps an analogy may also be drawn between the topological structure of bent, closed, or knotted hydrodynamical vortex tubes, in a semiturbulent situation, and the Copenhagen ("spaghetti") vacuum of interwoven flux tubes of QCD.[16] We do not know the answer to this, but point out that, among others, it is an interesting question to ask.

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APPENDIX

All the material of this appendix has been taken from Ref. 4. The eight, traceless SU(3) matrices λ_i satisfy the relations

$$\mathrm{Tr}[\lambda_i \lambda_j] = 2\delta_{ij},$$

$$[\lambda_i,\lambda_j] = 2if_{ijk}\lambda_k,$$

and

$$\{\lambda_i,\lambda_j\} = \tfrac{4}{3}\delta_{ij} + 2d_{ijk}\lambda_k.$$

An explicit representation of the λ_i may be written as

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda_2 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ \lambda_8 = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}.$$

Nonzero elements of f_{ijk} and d_{ijk} are given below. The f_{ijk} are odd under permutation of any two indices, while the d_{ijk} are even.

ijk	f_{ijk}	ijk	d_{ijk}
123	1	118	$1/\sqrt{3}$
147	1/2	146	1/2
156	$-1/2$	157	1/2
246	1/2	228	$1/\sqrt{3}$
257	1/2	247	$-1/2$
345	1/2	256	1/2
367	$-1/2$	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	1/2
678	$\sqrt{3}/2$	355	1/2
		366	$-1/2$
		377	$-1/2$
		448	$-1/(2\sqrt{3})$
		558	$-1/(2\sqrt{3})$
		668	$-1/(2\sqrt{3})$
		778	$-1/(2\sqrt{3})$
		888	$-1/\sqrt{3}$

[1]H. Aref and N. Pomphrey, Phys. Lett. A **78**, 297 (1980), and other papers quoted therein.

[2]E. S. Fradkin, Nucl. Phys. **76**, 588 (1966); E. S. Fradkin, U. Esposito, and S. Termini, Nuovo Cimento, Ser. I, **2**, 498 (1970).

[3]See, for example, G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge U. P., Cambridge, 1967), Chap. 5.

[4]It should be noted that only the symmetric part of $\partial_b v_a$ can enter into (4), or (2), as long as $\boldsymbol{\omega} = \nabla \times \mathbf{v}$, although for simplicity and possible application to other problems, we keep to the SU(3) forms of the text. But when approximations are made, one must be careful to consider quantities manifestly independent of the vorticity components $\psi_{2,5,7}$ of (43), such as Ω-growth estimates of Sec. IV, which really depend on the strain ϕ only. The quantity $\xi(t)$ of (60) then becomes $\widehat{G}\cdot\hat{\phi}$, where Φ is replaced by ϕ.

[5]A discussion of ordered exponentials can be found in any text on quantum field theory or many-body physics. One which contains, in addition, some useful functional methods is H. M. Fried, *Functional Methods and Models in Quantum Field Theory* (M.I.T. Press, Cambridge, MA, 1972).

[6]M. Gell-Mann and Y.Ne'eman, *The Eight-fold Way* (Benjamin, New York, 1964).

[7]See, e.g., P. Germain in *Les Houches Summer School Lectures*, edited by R. Balian and J.-L. Peube (Gordon and Breach, Paris, 1973).

[8]See, e.g., the discussion given by U. Frisch and R. Morf, Phys. Rev. A **23**, 2673 (1981), and other references quoted therein.

[9]These SU(3) techniques were used by H. M. Fried, Proceedings of the U.S.–France Scientific Seminar "Theoretical Aspects of QCD", Marseille, 1981; similar methods were used in an SU(2) context by H. F. Fried, Phys. Rev. D **16**, 1916 (1977). Of course, very many different methods have been used to approximate the effects of vortex stretching and the possible development of finite-time singularities, e.g., U. Frisch, in *Les Houches Summer School Lectures* (Gordon and Breach, Paris, 1981), and many references quoted therein.

[10]More precisely, G may have an imaginary component, $G \rightarrow G + iH$, where the components of G and H are real, and they multiply the real and imaginary Gell-Mann matrices, respectively. All the analysis which follows (62) refers only to the real part of the eigenvalues of $\boldsymbol{\lambda}\cdot(\mathbf{G} + i\mathbf{H})$, for that is the quantity entering into the crude estimates of enstrophy following (64).

[11]Starting from a scalar equation, similar to (44) with $\boldsymbol{\omega} = 0$, R. H. Kraichnan, [J. Fluid Mech. **64**, 737 (1974)] has extracted special forms of intermittency. It is, at present, an open and interesting question if anything as solid can be extracted from (59) for the NS problem.

[12]This is a big and, to our knowledge, unjustified assumption, which attempts to link the deterministic NS properties under discussion with quantities more properly defined in terms of statistical averages. By using only a single length scale, we envisage a situation in which certain regions of the fluid are locally dense with tightly coiled vortex rings; as the rings stretch, their radii decrease and they immediately coil and twist and could, as the analysis suggests, possibly fit into a smaller spatial region, of smaller length scale and higher vortex density. But there is no reason to treat this crude argument as anything more than an interesting perhaps oversimplified exercise in dimensional analysis. A survey of other, different approaches to the growth of enstrophy can be found in S. Orszag, in Ref. 7.

[13]$l(t) \sim (t^* - t)^{2/5}$ and $v(t) \sim (t^* - t)^{-6/5}$ produce a result with the same exponents as those found in the analysis of energy decay by P. G. Saffman, Phys. Fluids **10**, 1349 (1967).

[14]I. Proudman and W. H. Reid, Philos. Trans. R. Soc. London Ser. A **247**, 163 (1954). If it is assumed that energy *density* is conserved, then enstrophy density $\frac{1}{2}\omega^2 \sim [l(t)]^{-2}$, and the techniques leading to (66) will instead produce $\omega^2 \sim [l_0 - c\int_0^t dt'\,\xi(t')]^{-2}$, where c is a constant. If the function used to model $\xi(t)$ satisfies possibility (i), then $\omega^2 \sim (t^* - t)^{-2}$, which is the same as that found in the analysis of this reference. More recently, a similar growth of vorticity has been found to follow from a model of the NS equations by P. Vieillefosse, J. Phys. (Paris) **43**, 837 (1982). Other work on this subject may be found in J. Léorat, Thèse de Doctorat, Université de Paris -VII, 1975.

[15]G. 't Hooft, Nucl. Phys. B **72**, 461 (1974). A very modern reference is C-I. Tan, Zheng Xi-Te, and Chen Tian-lun, "Reduced Large N Lattice Theory Without Spontaneous Symmetry Breakdown," Phys. Letters (to be published).

[16]H. B. Nielsen in *Proceedings of the 3rd Adriatic Summer Meeting on Particle Physics*, Dubrovnik, 1980, edited by I. Andric, I. Padic, and N. Zorko (North-Holland, Amsterdam (1980). The original paper is H. B. Neilson and P. Olesen, Nucl. Phys. B **160**, 380 (1979).