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# Downwelling irradiance fluctuations in the small-angle approximation

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## ABSTRACT

Mean and rms fluctuations of downwelling solar irradiance below a rough ocean surface have been modelled using the small-angle approximation for the in-water radiance distribution. For comparison, a ray-trace approach has also been used to calculate the fluctuation statistics in the limit of no scattering. The mean irradiance decays rapidly near the surface, and at a sufficiently large depth (determined by the water optical properties) is within a few percent of the irradiance for a flat ocean surface. Although rms fluctuations decay with depth as well, their magnitude relative to the mean irradiance reaches a nonzero asymptotic value which depends on the sun position, optical properties, and surface roughness.

## 1 INTRODUCTION

Waves on the ocean surface strongly affect the natural light field just below the surface by inducing irradiance fluctuations in space and time. The primary mechanism for this is focussing and defocussing of incident light by wave-induced surface curvature. When large swells are present fluctuations can also occur through variations in the attenuating path length along the surface of the waves.

Several researchers have examined the biological impact of irradiance fluctuations in the near surface photic zone[1], documenting the temporal spectrum of surface waves and irradiance fluctuations at several depths. Dera and Olszewski[2], and later Snyder and Dera[3], measured time series of downwelling irradiance fluctuations at several depths in gulf waters 7 m deep, generating mean and variance statistics as a function of depth. Siegel and Dickey[4] reported fluctuation statistics to a depth of 50 m in open ocean. In terms of optical depth (at wavelengths of 400-500 nm) these works examined the light field to a depth of about one diffuse attenuation length. Very little is known about the fate of irradiance fluctuations at greater depths.

The aim of this paper is to provide a theoretical description of the decay of downwelling irradiance fluctuations at depths ranging from one beam attenuation length to asymptotic depths. Fluctuations are characterized below by the mean and variance of the downwelling irradiance. For this purpose the small-angle approximation described by Tessendorf[5] is used to model the downwelling irradiance in an unstratified ocean with a clear sky and only solar incident lighting. This approach is useful because

1. This version of the small-angle approximation compared reasonably well[6] with data collected in a lake with a relatively flat surface[7].
2. The small-angle approximation allows analytical manipulations, providing explicit expressions for irradiance statistics. This aids in simplifying the description, while retaining the dominant physical features of the light field.
3. In the spirit of the small-angle approximation, the incident angle dependence of the Fresnel transmission coefficient is ignored, further simplifying the description of the light field.

Because of the small-angle approximation however, the near surface irradiance field cannot be obtained and the accuracy of the approximation is severely degraded at solar angles greater than about 45° from the vertical. The small-angle approximation is reviewed in section 3 and the mean and variance of the downwelling irradiance are calculated.

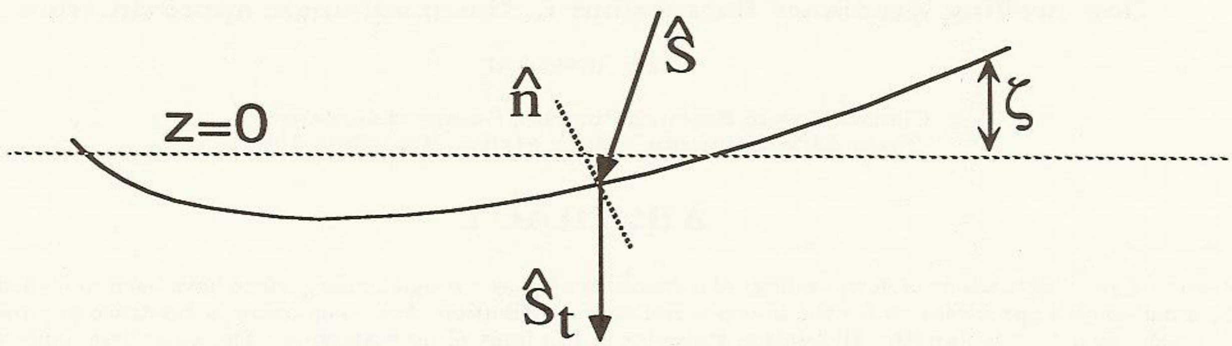


Figure 1: Geometry of the refraction of incoming light through the ocean surface.

Before applying the small-angle approximation however, the mean and variance of the downwelling irradiance are calculated in section 2 in the limit of no scattering. The approach there is an improvement of the ray-trace model described by Snyder and Dera[3], and is presented to contrast with the results when scattering is present.

In both sections 2 and 3, irradiance statistics are induced by the statistics of surface slope. The amplitude of surface waves is assumed to be small compared to the depths of interest, so that amplitude-induced fluctuations are ignored. The magnitude of surface slopes is also assumed to be small, which is consistent with measured surface slope statistics in low sea states. In fact the auto statistics of surface slopes is accurately modeled by a gaussian form

$$P(\vec{\epsilon}) = \frac{\det \Lambda^{-1/2}}{2\pi} \exp\left\{-\frac{1}{2}\vec{\epsilon}\cdot\Lambda^{-1}\cdot\vec{\epsilon}\right\} ,$$

where $P$ is the probability density of surface slope $\vec{\epsilon}$ and $\Lambda_{ij} = \langle\epsilon_i\epsilon_j\rangle$ is the slope autocorrelation matrix. Preisendorfer and Mobley[10] used the upwind and crosswind parameterization

$$\begin{aligned} \hat{U}\cdot\Lambda\cdot\hat{U} &= \lambda_u = 0.0032U \\ \hat{U}_\perp\cdot\Lambda\cdot\hat{U}_\perp &= \lambda_c = 0.0019U \end{aligned}$$

where $\hat{U}$ is the horizontal direction of winds just above the ocean surface, $\hat{U}_\perp$ is the horizontal direction perpendicular to $\hat{U}$, and $U$ is the windspeed in m/s. This formulation of surface statistics is used below as a convenient way of examining a range of surface slope statistics. It is assumed that the correlation distance of surface slopes is relatively large compared to an optical depth, so that the autocorrelation statistics of surface slopes is the only necessary input to the calcuation of the irradiance variance. To the extent that this assumption is not valid, the calculated variances below can be considered upper bounds.

## 2 NO SCATTERING LIMIT

In the limit of no scattering ($b \longrightarrow 0$), the irradiance field can be modelled by following the paths of rays of light after transmission through the surface. The irradiance at any depth is proportional to the concentration of rays on an imaginary surface of unit length, with each ray assigned a weight by absorption along its path.

### 2.1 Refraction through the ocean surface

Figure 1 depicts refraction of incoming solar light through the ocean surface. The incident direction $\hat{S}$ is the same at all points of the surface, but the transmitted direction $\hat{S}_t$ depends on the local normal $\hat{n}$ through Snell's

Law:

$$\hat{S}_t = \frac{n_a}{n_w}\left(\hat{S} - \hat{n}(\hat{S}\cdot\hat{n})\right) + \hat{n}\left\{1 - \left(\frac{n_a}{n_w}\right)^2 |\hat{S}\times\hat{n}|^2\right\}^{1/2} ,$$

where $n_a$, $n_w$ are the indices of refraction for air and water ($n_w/n_a \approx 1.34$). The surface normal depends on the slope of the surface at that point:

$$\hat{n} = \frac{\hat{z} - \vec{\epsilon}}{\sqrt{1+\epsilon^2}} .$$

A ray incident on the ocean surface at the horizontal position $\vec{x}_0$ and surface wave height $z = \zeta(\vec{x}_0)$ has the horizontal position $\vec{x}(z)$ at depth $z$ given by

$$\vec{x}(z) = \vec{x}_0 + \vec{m}(\vec{x}_0)(z - \zeta(\vec{x}_0)) ,$$

where

$$\vec{m} = \frac{\hat{S}_t - \hat{z}(\hat{z}\cdot\hat{S}_t)}{\hat{z}\cdot\hat{S}_t} .$$

Each ray is attenuated by the factor $\exp\{-ad\}$, where $a$ is the absorption coefficient and $d$ is the distance travelled by the ray to the position $\vec{x}(z)$:

$$\begin{aligned} d^2 &= (z - \zeta(\vec{x}_0))^2 + |\vec{x} - \vec{x}_0|^2 \\ &= (z - \zeta(\vec{x}_0))^2\,(1 + |\vec{m}(\vec{x}_0)|^2) . \end{aligned}$$

The downwelling irradiance at the point $(\vec{x}, z)$ is the sum of the contributions of all rays which reach that point:

$$I(\vec{x}, z) = I_0 \int d^2x_0 \;\exp\{-ad\}\;\delta\left[\vec{x} - \vec{x}_0 - \vec{m}(\vec{x}_0)(z - \zeta(\vec{x}_0))\right] .$$

Note that in the limit of a flat surface, this is

$$I_{\text{flat}}(z) = I_0 \exp\left\{-az\left[1 + \left(\frac{n_a}{n_w}\right)^2 \tan^2\theta_s\right]^{1/2}\right\} ,$$

where $\theta_s$ is the angle of the sun from the vertical ($\cos\theta_s = \hat{z}\cdot\hat{S}$).

### 2.2 Irradiance statistics

This formulation follows closely the model used by Snyder and Dera[3], with the exception that the Fresnel transmission factor is ignored here. In computing the irradiance statistics, Snyder and Dera perturbatively expanded the irradiance in terms of the surface slope, obtaining a mean irradiance equal to that for the flat surface case, and a variance proportional to the variance of the surface slope. The irradiance however significantly depends on the surface slope, and a more accurate method is to expand the distance $d$ up to quadratically in the surface slope when evaluating the averages. This is the approach used below.

As stated in the introduction, we are interested in depths $z$ much greater that the surface amplitudes, so we ignore the presence of $\zeta$ in $I$. The mean downwelling irradiance is

$$\langle I(z)\rangle = I_0 \int d^2x_0 \int d^2\epsilon\; P(\vec{\epsilon})\exp\{-ad\}\;\delta\left[\vec{x} - \vec{x}_0 - \vec{m}(\vec{x}_0)(z - \zeta(\vec{x}_0))\right] .$$

By expanding in small surface slopes $\vec{\epsilon}$, all of the integrations become gaussian, with the result

$$\langle I(z)\rangle = I_{\text{flat}}(z)\;\det\left(1 + \frac{az}{\{1+S_0^2\}^{1/2}}\Lambda\cdot M\cdot R\cdot M\right)^{-1/2} \exp\left\{\frac{1}{2}\frac{(az)^2}{1+S_0^2}(\vec{S}_0\cdot M\cdot Q\cdot M\cdot\vec{S}_0)\right\} , \qquad (1)$$

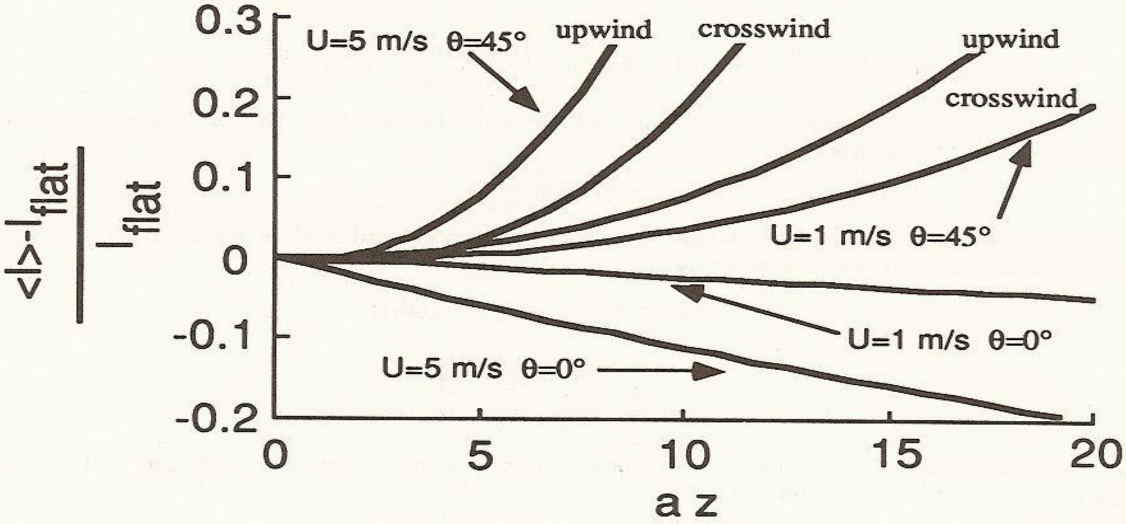


Figure 2: Relative difference between the mean irradiance from a rough ocean surface and the irradiance from a flat surface, in the limit of no scattering, for several wind speeds and sun positions.

where

$$\vec{S}_0 = \hat{S} - \hat{z}(\hat{z} \cdot \hat{S}) \ ,$$

and the matrices $R$, $M$, and $Q$ are

$$R = 1 - 2\frac{\vec{S}_0\vec{S}_0}{1+S_0^2} \ ,$$

$$M = 1 + \frac{n_a}{n_w}\left(1 - \frac{(\hat{z}\cdot\hat{S})(n_a/n_w)}{\{1+(n_a/n_w)^2\sin^2\theta_s\}^{1/2}}\right)\frac{\hat{z}\hat{S}}{\{1+(n_a/n_w)^2\sin^2\theta_s\}^{1/2}} \ ,$$

$$Q^{-1} = \Lambda^{-1} + \frac{az}{\{1+S_0^2\}^{1/2}} M \cdot R \cdot M \ .$$

Figure 2 shows the relative difference

$$\delta\iota \equiv \frac{\langle I \rangle - I_{\rm flat}}{I_{\rm flat}}$$

between the flat and rough surface mean irradiances for several sun positions and wind speeds. For the sun directly overhead at $\theta_s = 0$ the relative difference asymptotically approaches -1 as

$$\delta\iota \rightarrow -1 + O\left(\frac{1}{az}\right) \quad (az \gg 1)$$

because the rough surface effectively diffuses the incoming solar light. This diffusion reduces the magnitude of the mean irradiance below the flat surface irradiance level by the determinant factor in equation 1.

When the sun is at an angle $\theta_s \neq 0$, the relative difference follows the same behavior at depths shallower than $az \sim 1/(\lambda_c^{1/2}\sin\theta_s)$, at which point the difference grows exponentially. In this case the surface roughness causes some facets of the ocean surface to face the sun more than for a flat ocean. This concentrates the incident sunlight and decreases the attenuation compared to the flat ocean case.

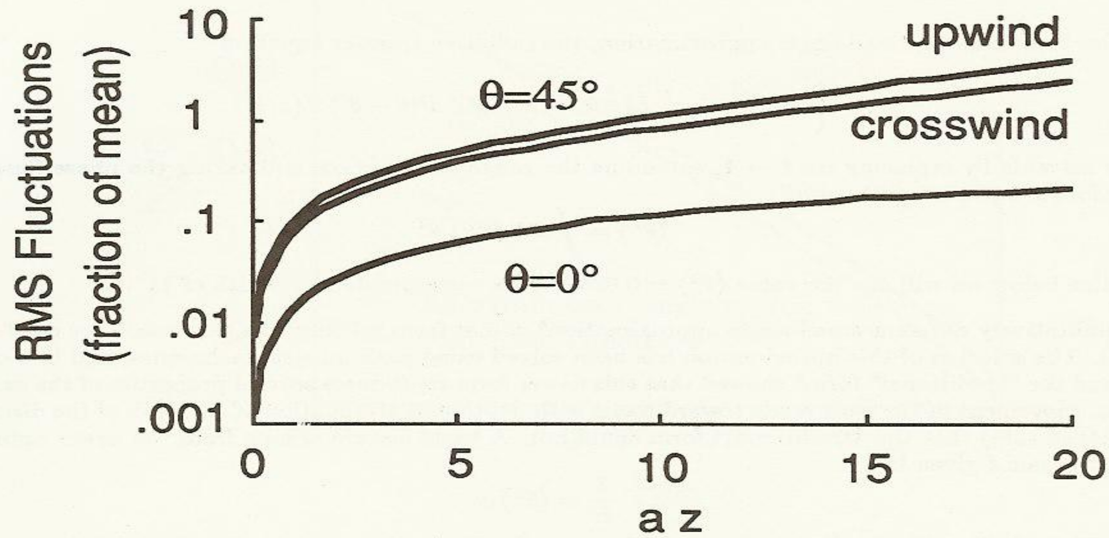


Figure 3: RMS fluctuations of the downwelling irradiance as a function of depth for the sun at $\theta_s = 0°, 45°$, in the limit of no scattering. The wind speed is 5 m/s.

A similar evaluation for the mean square gives

$$\begin{aligned}\langle I^2(z)\rangle &= I_0^2 \int d^2x_0 \int d^2\epsilon\; P(\vec{\epsilon})\; \exp\{-2ad\}\; \delta\left[\vec{x} - \vec{x}_0 - \vec{m}(\vec{x}_0)(z - \zeta(\vec{x}_0))\right] \\ &= \langle I(z)\rangle^2 \det\left(1 + \frac{2az}{\sqrt{1+S_0^2}} M\cdot R\cdot M\right)^{-1/2} \det\left(1 + \frac{az}{\sqrt{1+S_0^2}} M\cdot R\cdot M\right) \\ &\times \exp\left\{\frac{(az)^2}{1+S_0^2}(\vec{S}_0\cdot M\cdot W\cdot M\cdot \vec{S}_0)\right\},\end{aligned}$$

where now the matrix $W$ is

$$W = 2\left[\Lambda^{-1} + \frac{2az}{\{1+S_0^2\}^{1/2}} M\cdot R\cdot M\right]^{-1} - Q\,.$$

For large $az$, the relative rms fluctuations

$$\delta^2\iota \equiv \left\{\frac{\langle I^2(z)\rangle - \langle I(z)\rangle^2}{\langle I(z)\rangle^2}\right\}^{1/2}$$

grows as

$$\delta^2\iota \sim (az)^{1/2} \quad (az \gg 1)\,.$$

Figure 3 shows the behavior of $\delta^2\iota$ as a function of depth for several sun positions. This result is inconistent with measured rms fluctuations, which decay with depth relative to the mean, although there can be a shallow region in the first few meters below the surface where the fluctuations initially build up[2,3,4].

## 3 SMALL-ANGLE APPROXIMATION

In the "traditional" small-angle approximation, the radiative transfer equation

$$\left\{\cos\theta \frac{\partial}{\partial z} + c\right\} L(z,\theta) = b \int d\Omega'\ P(\theta - \theta')\ L(z,\theta')$$

is made solvable by replacing $\cos\theta \rightarrow 1$, extending the range of $\theta$ to $\pm\infty$, and taking the phase function to be sharply forward-peaked with width

$$\langle\theta^2\rangle = \int d\Omega\ P(\theta)\ \theta^2\ .$$

In practice below we will use the value $\langle\theta^2\rangle = 0.035$, which corresponds to a width of 11°.

A qualitatively different small-angle approximation[5] comes from relaxing the expression for $\cos\theta$ to $\cos\theta \rightarrow 1 - \theta^2/2$. The solution of this aproximation has been solved using path integral techniques, and in a comparison of this and the "traditional" form[6] showed that this newer form reproduces several properties of the experimental data (i.e. movement of the peak angle toward nadir with depth and attenuation of the peak of the distribution at the observed rate) that the "traditional" form could not. A basic feature arising from the newer approximation is a length scale $\ell$ given by

$$\frac{1}{\ell^2} = \langle\theta^2\rangle\, ab\ .$$

This length scale is a diffusion-limiting scale, in the sense that at depths $z < \ell$ the distribution behaves similarly to the "traditional" solution and broadens with depth in a diffusive way, but at depths $z \geq \ell$ the diffusive evolution of the distribution is cutoff and the asymptotic distribution emerges. In addition, while the "traditional" small-angle approximation gives a diffuse attenuation coefficient of $K_d = a$, this newer form gives

$$K_d = a + \frac{1}{\ell}\ ,$$

which is in qualitatively better agreement with data and other models than the "traditional" result. For example, figure 4 shows $K_d/c$ as a function of $b/c$ from this small-angle approximation, and from the relation

$$\frac{K_d}{c} = \left\{\beta\left(1 - \frac{b}{c}\right)\right\}^{b/2c}\ ,$$

suggested by Timofeeva[8]. The value $\beta = 0.3211$ was obtained by Tanis et.al.[9] for the NUC2200 phase function.

In a general radiative transfer problem the radiance $L(z, \vec{x}, \hat{s})$ at depth $z$, horizontal position $\vec{x}$, and direction $\hat{s}$ is given in terms of the distribution on the surface and an evolution operator $G$:

$$L(z, \vec{x}, \hat{s}) = \int d\Omega' \int d^2x_0\ G(z, \vec{x}, \hat{s}; \vec{x}_0, \hat{s}')\ L(0, \vec{x}_0, \hat{s}')\ .$$

For the solar distribution incident on the rough surface, the initial distribution is

$$L(0, \vec{x}_0, \hat{s}') = I_0\ \delta\left(\hat{s}' - \hat{S}_t(\vec{x}_0)\right)$$

Surface height elevations are ignored here, so that the surface is modelled as flat but with spatially varying slope $\vec{\epsilon}(\vec{x}_0)$. The irradiance at depth $z$ and position $\vec{x}$ for a particular realization of the ocean surface is

$$I(\vec{x}, z) = I_0 \int d\Omega \int d^2x_0\ G\left(z, \vec{x}, \hat{s}; \vec{x}_0, \hat{S}_t(\vec{x}_0)\right)\ .$$

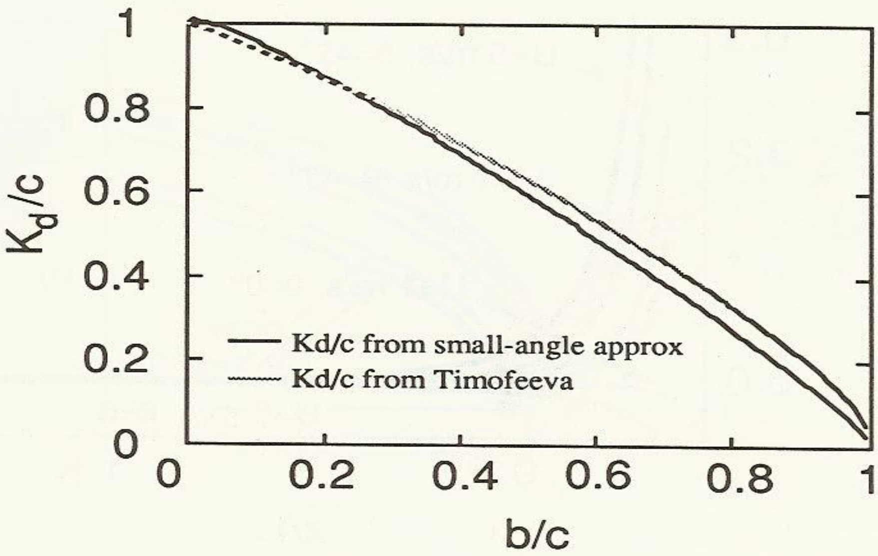


Figure 4: Plot of $K_d/c$ *vs* $b/c$ for the small-angle approximation and the form suggested by Timofeeva.

The small-angle approximation generates an explicit expression for the evolution operator $G$. Using a procedure similar to that in section 2 for expanding in surface slopes, the integrals in space, direction, and slope can be evaluated. The quantities $\delta\iota$ and $\delta^2\iota$ defined above become

$$\begin{aligned}\delta\iota &= \det\left(1+\alpha^2\omega(\xi)\Lambda\right)^{-1/2}\exp\left\{\frac{1}{2}\omega^2(\xi)\vec{S}_0\cdot\Lambda\cdot\left(1+\alpha^2\omega(\xi)\Lambda\right)^{-1}\cdot\vec{S}_0\right\}-1\\ \delta^2\iota &= \left\{\det\left(1+2\alpha^2\omega(\xi)\Lambda\right)^{-1/2}\det\left(1+\alpha^2\omega(\xi)\Lambda\right)\exp\left\{-\omega(\xi)\vec{S}_0\cdot H\cdot\vec{S}_0\right\}-1\right\}^{1/2},\end{aligned}$$

where $\xi = z/\ell$ and

$$\begin{aligned}\alpha &= \frac{n_a}{n_w}(\hat{S}\cdot\hat{z})+\left\{1-\left(\frac{n_a}{n_w}\right)^2|\hat{S}\times\hat{z}|^2\right\}^{1/2},\\ \omega(\xi) &= \frac{1}{2\langle\theta^2\rangle b\ell}\frac{\sinh(2\xi)}{\sinh^2(\xi)}\left\{1+4\left(\frac{\sinh(\xi)}{\sinh(2\xi)}\right)^2\right\},\\ H &= \left(1+2\alpha^2\omega(\xi)\Lambda\right)^{-1}-\left(1+\alpha^2\omega(\xi)\Lambda\right)^{-1}.\end{aligned}$$

Notice that these quantities depend only on $\langle\theta^2\rangle$, $z/\ell$, $b/c$, the sun position $\hat{S}$, and the windspeed $U$.

Figure 5 shows $\delta\iota(z)$ for several solar angles and wind speeds, and figure 6 similarly shows $\delta^2\iota(z)$. Below $\xi \sim 0.5$ the light field has broadened sufficiently to lose nearly all sensitivity to the rough ocean surface, and the mean irradiance is within a few percent of the value from a flat ocean surface. This is potentially a very useful result, because it implies that models need not incorporate rough surfaces as long as they are not intended for the top layer of the ocean. The sharp fall of $\delta\iota$ from the surface in figure 5 is at least partly due to the shortcomings fo the small-angle approximation, although sophisticated numerical models show similar behavior after a brief near-surface build up of diffusive light[11]. Although the rms fluctuations also decay rapidly near the surface, figure 6 clearly shows the $\delta^2\iota$ asymptotically approaches a non-zero limit at great depth. In fact both $\delta\iota$

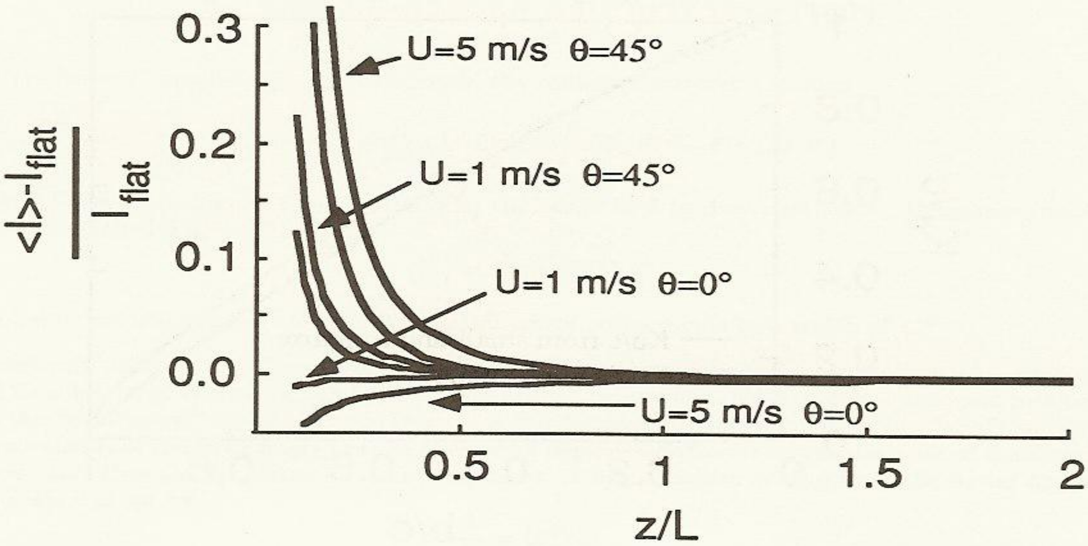


Figure 5: Mean irradiance as a function of depth, expressed as a fraction of the irradiance from a flat ocean surface. The solar positions are $\theta_s = 0°$ and $45°$, both upwind and crosswind.

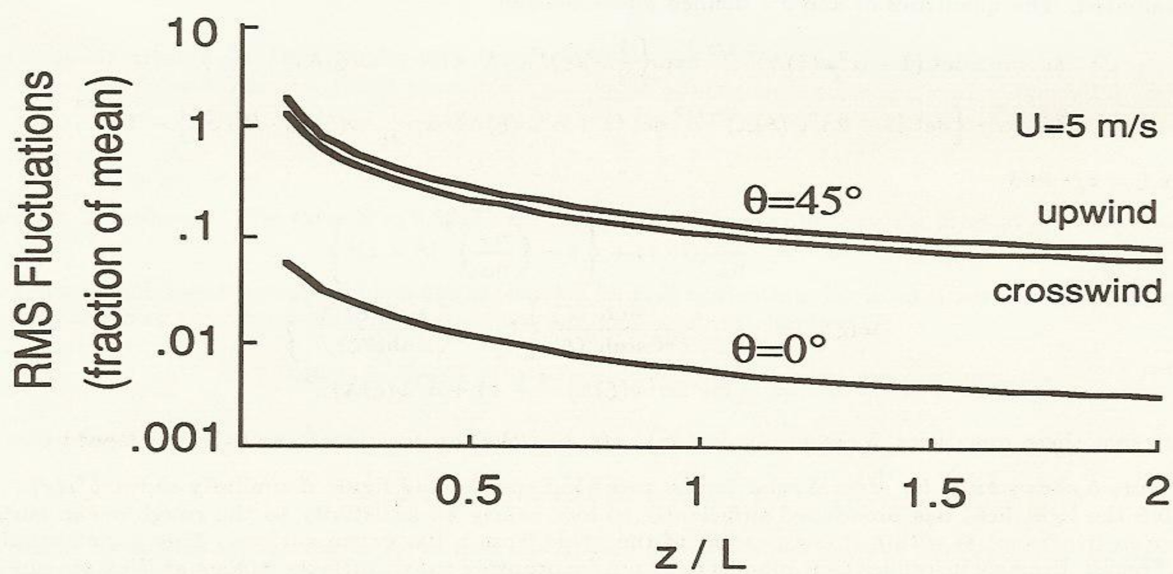


Figure 6: Rms fluctuations in the downwelling irradiance, expressed as a fraction of the mean irradiance.

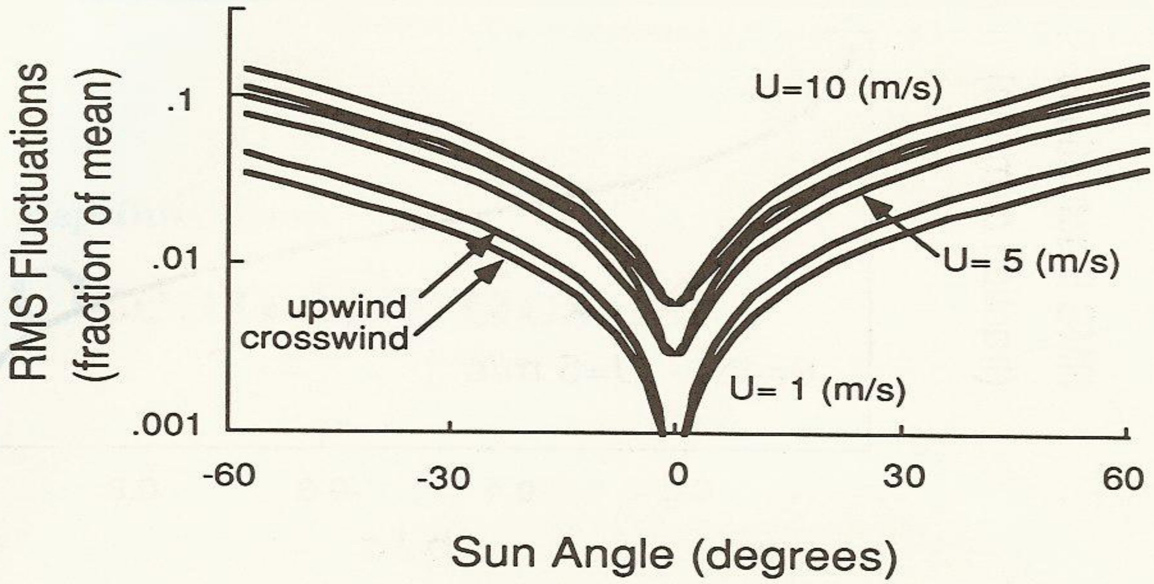


Figure 7: Dependence of asymptotic rms fluctuations on sun position.

and $\delta^2\iota$ approach nonzero limits because $\omega(\xi)$ behaves asymptotically as

$$\omega(\xi \to \infty) = \frac{1}{\langle\theta^2\rangle\, b\ell} .$$

The asymptotic values of $\delta\iota$ and $\delta^2\iota$ depend only on $\langle\theta^2\rangle$, $b/c$, the sun position, and the wind speed. Figure 7 shows the asymptotic rms fluctuations as a function of sun position for several wind speed and $b/c = 0.7$. If the sun is not overhead we can reasonably expect in many situations to have rms fluctuations at 5% – 15% of the mean irradiance regardless of depth. Figure 8 shows asymptotic rms fluctuations as a function of $b/c$ for a wind speed of 5 m/s and the sun upwind at 45°.

## 4 CONCLUSIONS

From a comparison of figure 2 with 5, and figure 3 with 6, it is clear that scattering severely affects fluctuations in the underwater light field. Qualitatively this statement can be expected to be true without a detail theoretical investigation. However, what may not have been expected is the presence of irradiance fluctuations with a magnitude that is a significant fraction of the mean level even at great depths. It would be interesting to see if these fluctuations exist in experimental data and in numerical models. If they are found to exist, they may be exploitable in radiative transfer inversion schemes. Their existence might also raise the issue of whether they play a role in the biological activity of the deep ocean.

## 5 ACKNOWLEDGEMENTS

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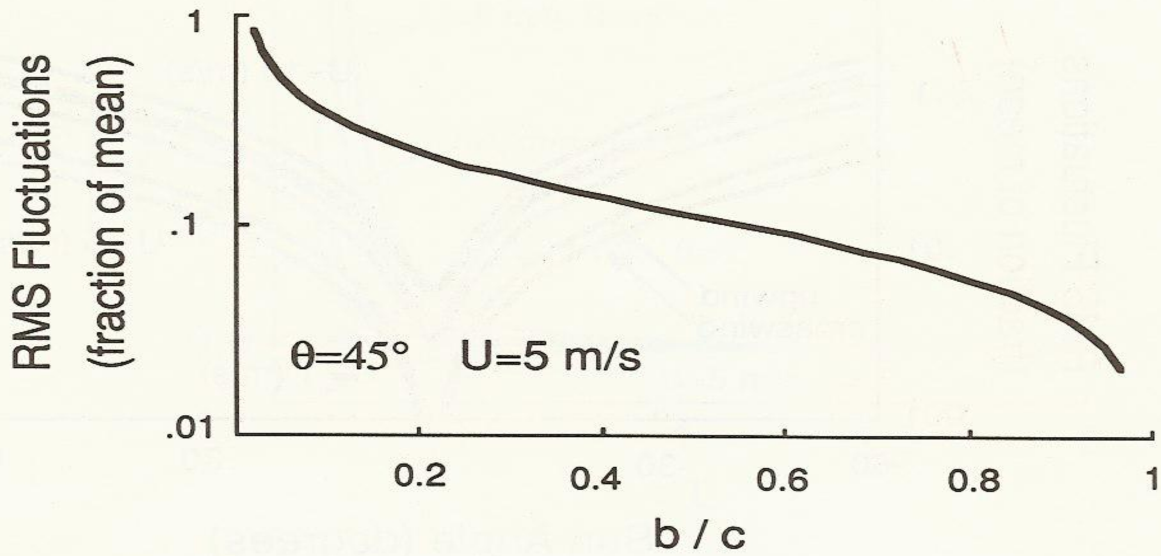


Figure 8: Dependence of asymptotic rms fluctuations on the single scatter albedo $b/c$, for a sun position upwind at 45° and a windspeed of 5 m/s.

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