Implementation of Curved Strands I: Frenet-Serret Framework

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1 Introduction

This paper presents a solution to the problem of simulating curved hair strands and their dynamic evolution in the presence of forces. This problem has been approached in several ways in the recent past. The scheme we outline here, to the best of our knowledge, is unique in the computer graphics field. Of course, the techniques and results we discuss are previously known in other mathematical fields. Because of this, a large portion of this paper is devoted to deriving the theoretical and numerical details of the implementation.

Aside from the scheme we adopt here, there are two primary branches of activity for curved hair representation and dynamical evolution. The first of these models a hair strand as a sequence of rods linked end-to-end. The rods are free to pivot on the linked nodes, although under the restoring forces of imaginary springs. Dynamics in this model is implemented at the nodes, causing a node to pivot about the node on the opposite end of the rod, and restricted by the stiffness of the rod to bending, stretching or twisting. In this approach, modeling smoothly and complex shapes requires many nodes, and generating smooth stable dynamics can require more still. To reduce the number of nodes, this method can be modified so that between nodes the shape is modeled as a spline. The dynamics however, must still be applied to nodes.

The second branch models the hair strand as the shape of the path traversed by a particle under some force. With access to a good particle dynamics package, this approach can generate strands with complex shapes. Since hair shape is the result of particle dynamics, hair dynamics is the result of time-evolution of the force(s) that shape the hair. This complicates the dynamical scheme considerably, since the standard dynamical relationship $F = ma$ is not the basis of the strand dynamics.

The alternative scheme we present below solves the problems of representing smooth-yet-complex hair strands with relatively little data, with dynamical evolution satisfying $F = ma$. The cost for this is the adoption of an exact parametric representation in terms of curvature and torsion (torsion and helicity are related); and dynamical equations that, while arising from $F = ma$, are much more complex than for rods and nodes. Details of the dynamical algorithm are contained in a companion paper[1]

Our scheme relies entirely on the classic Frenet[2]-Serret[3] Apparatus of space curves, and the consequent Fundamental Theorem of Curves. From these devices, we can completely characterize any conceivable (differentiable) curve in terms of the curvature and torsion along the curve. For convenience, the torsion is sometimes replaced in the representation by the helicity.

2 Frenet-Serret Apparatus for Space Curves

Generally, we characterize a space curve as a set of connected points in space, linked by a single parameter. Thus $\vec{x}(t)$ with $a \leq t \leq b$ is a space curve parameterized by t. This description is general enough to handle spline-based data, knots, and virtually every circumstance in a computer graphics application. There are no restrictions on the range of t or even its units. However, it is convenient and very useful to create a parameterization based on arclength s. At a point t on the curve, the arclength s is

$$s = \int_a^t dt' \left| \frac{d\vec{x}(t')}{dt'} \right| \tag{1}$$

Because the integrand is positive definite, there is a unique 1-1 mapping between t and the arclength s. We can therefore always think of the space curve as parameterized by arclength without any risk of trouble. From this point on, we will do just that.

Once parameterized by arclength, the tangent $\hat{T}(s) = d\vec{x}(s)/ds$ is in fact a unit vector. In addition, $d\hat{T}(s)/ds$ is perpendicular to $\hat{T}$ with magnitude $\kappa(s) = |d\hat{T}(s)/ds| \geq 0$, which is the curvature of the space curve.

Defining the unit vector $\hat{N}$ by

$$\frac{d\hat{T}(s)}{ds} = \kappa(s)\hat{N}(s) \ , \tag{2}$$

and a third unit vector $\hat{B}(s) = \hat{T}(s) \times \hat{N}(s)$, this set constitutes a complete orthonormal basis that moves and aligns with the space curve in a natural way. The set $\{\hat{T}(s), \hat{N}(s), \hat{B}(s)\}$ is called the *Frenet-Serret Basis*[4].

Because the Frenet-Serret basis is complete, we can represent the derivatives of the unit vectors in terms of the same vectors. We already have the derivative of $\hat{T}(s)$. The derivative of $\hat{N}(s)$ must be a linear combination of $\hat{T}(s)$ and $\hat{B}(s)$. From orthogonality,

$$\hat{T} \cdot \frac{d\hat{N}(s)}{ds} = -\hat{N} \cdot \frac{d\hat{T}(s)}{ds} = -\kappa(s) \ . \tag{3}$$

[1]J. Tessendorf and D.D. Weston, "Implementation of Curved Strands II: Dynamics," Cinesite Digital Studios, 1998.

[2]F. Frenet, "Sur les courbes à double courbure," Thèse, Toulouse, 1847. Abstract in *J. de Math.* **17**, 1852.

[3]J. A. Serret, "Sur quelques formules relatives à la théorie des courbes à double courbure," *J. de Math.* **16**, 1851.

[4]For a basic reference, try *Elements of Differential Geometry*, by Richard S. Millman and George D. Parker, Prentice-Hall, 1977.

The torsion $\tau(s)$ is defined as the coeffient for $\hat{B}(s)$:

$$\frac{d\hat{N}(s)}{ds} = -\kappa(s)\hat{T}(s) + \tau(s)\hat{B}(s)\ . \tag{4}$$

From this result and the definition of $\hat{B}$ we conclude

$$\frac{d\hat{B}(s)}{ds} = -\tau(s)\hat{N}(s)\ . \tag{5}$$

The set $\{\hat{T}(s), \hat{N}(s), \hat{B}(s), \kappa(s), \tau(s)\}$ is called the *Frenet-Serret Apparatus*. It has a very important role in the characterization of space curves because (1) it arises naturally from the curve, (2) is invariant under translations of the curve through space, (3) the curvature and torsion are invariant under rotations of the curve, and (4) the Fundamental Theorem of Curves assures us that the Frenet-Serret Apparatus is all the information we need or can get.

The Fundamental Theorem of Curves also tells us that any space curve is completely characterized by its torsion, curvature, initial point in space, and the Frenet-Serret basis at that initial point. For the hair strand application here, the internal representation of any curved hair will be in terms of the curvature and torsion. At any root position, the groom vector and surface tangent are used to construct the Frenet-Serret basis at the root, and the Fundamental Theorem of Curves generates the curved strand at all other points.

3 General Space Curve Representation

A general, closed-form solution for a space curve is not available for arbitrary curvature and torsion. Complete solutions do exist for some special cases, which we will explore in this section. First, we look at the general problem, and build a formal solution in terms of ordered-exponentials. Although this is a formal solution, it can be exploited numerically, and serves to define the conditions of validity of the numerical discretization. The numerical version is explored in the next section.

Throughout this section we will use the helicity $h(s) = \tau(s)/\kappa(s)$. Regions of a curve in which $\kappa = 0$ are straight line segments, for which it is reasonable to define $h = 0$ and $\tau = 0$. In terms of helicity, the equations for the Frenet-Serret apparatus are

$$\frac{d\hat{T}(s)}{ds} = \kappa(s)\hat{N}(s) \tag{6}$$

$$\frac{d\hat{N}(s)}{ds} = \kappa(s)\left\{-\hat{T}(s) + h(s)\hat{B}(s)\right\} \tag{7}$$

$$\frac{d\hat{B}(s)}{ds} = -\kappa(s)h(s)\hat{N}(s) \tag{8}$$

which should be combined with the equation for the curve parameterization

$$\frac{d\vec{x}(s)}{ds} = \hat{T}(s)\ . \tag{9}$$

These equations constitute the complete set to place hairs on a surface. For each hair, the root position $\vec{x}(0)$ and orientation (grooming) at the root $\{\hat{T}(0), \hat{N}(0), \hat{B}(0)\}$ is combined with the curvature and helicity for the entire length of the strand. The actual space curve of the hair strand is then computed by solving this coupled set of first order differential equations.

One method for implementing artistic control over curvature and torsion is the use of guide hairs. After an artist lays out a set of guide hairs that are positioned such that an ordering in space can be defined, the curvature and torsion for each guide hair can be extracted. As hairs are created and placed, the guide hair properties are interpolated. Interpolation can take place at the level of curvature and torsion, or at the level of the solution for the space curve.

In the remainder of this section, we look at two solutions for the Frenet-Serret apparatus. The first is a general solution that is formal and somewhat abstract, but is amenable to a numerical implementation. The second is a complete solution under the simplifying assumption that the helicity is constant.

3.1 Ordered-Exponential Representation

To create the ordered-exponential solution, we first note that the differential equations for the Frenet-Serret basis can be put in a matrix form:

$$\frac{d\mathbf{F}(s)}{ds} = \mathbf{Q}(s) \cdot \mathbf{F}(s) \; . \tag{10}$$

where the matrix $\mathbf{Q}(s)$ is

$$\mathbf{Q}(s) = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \tag{11}$$

and $\mathbf{F}(s)$ is a vector with the Frenet-Serret basis as components:

$$\mathbf{F}(s) = \begin{bmatrix} \hat{T}(s) \\ \hat{N}(s) \\ \hat{B}(s) \end{bmatrix} \; . \tag{12}$$

The general form of equation 10 suggests an exponential solution. However, $\mathbf{Q}$ is non-commutative, i.e. if $\mathbf{Q}(s) \neq \mathbf{Q}(s')$, then $\mathbf{Q}(s) \cdot \mathbf{Q}(s') \neq \mathbf{Q}(s') \cdot \mathbf{Q}(s)$. Consequently, the ordering of the matrix in the solution is critically important, and the exponential-like solution is called an ordered-exponential, with the form

$$\mathbf{F}(s) = \left(\exp\left\{ \int_0^s ds' \, \mathbf{Q}(s') \right\} \right)_+ \cdot \; \mathbf{F}(0) \; . \tag{13}$$

The meaning of the ordered exponential comes from the Taylor expansion of equation 10, keeping careful track of the matrix order and commutation issues:

$$U(s) \;\; \equiv \;\; \left(\exp\left\{ \int_0^s ds' \, \mathbf{Q}(s') \right\} \right)_+$$

$$= \; 1 \; + \; \int_0^s ds' \mathbf{Q}(s') + \int_0^s ds' \int_0^{s'} ds'' \mathbf{Q}(s') \cdot \mathbf{Q}(s'')$$

$$+ \; \int_0^s ds' \int_0^{s'} ds'' \int_0^{s''} ds''' \mathbf{Q}(s') \cdot \mathbf{Q}(s'') \cdot \mathbf{Q}(s''') + ...$$

The section on numerical implementation presents an alternate expression for the ordered-exponential, given a discretizing scheme.

In placing a hair on a surface, with hair shape based on interpolating a few guide hairs, there are two possible schemes for interpolation: (1) interpolate $\kappa(s)$ and $\tau(s)$ from the guide hairs to a particular hair, construct the ordered exponential for the hair, construct $\mathbf{F}(s)$, and finally construct the space curve; or (2) construct the ordered-exponential for each separate guide hair, interpolate the ordered exponentials, construct $\mathbf{F}(s)$, and construct the space curve. The second method is faster because fewer ordered exponentials are computed, but the first method is potentially smoother and may have fewer artifacts.

There is a very important property belonging to the ordered-exponential U: It is an orthogonal transformation matris. This can be seen on two levels. First, it is required to be orthogonal, because the components of $\mathbf{F}$ are the complete orthonormal unit vectors of the Frenet-Serret basis. For any arbitrary s, it must always be true that

$$\left|\hat{T}(s)\right|^2 + \left|\hat{N}(s)\right|^2 + \left|\hat{B}(s)\right|^2 = \mathbf{F}^T(s) \cdot \mathbf{F}(s) = \mathbf{F}^T(0) \cdot \mathbf{F}(0) \qquad (14)$$

which can only work out if U is orthogonal. Second, by explicit construction, $\mathbf{Q}^T = -\mathbf{Q}$, so that $U^T = U^{-1}$, which is the definition of an orthogonal transformation matrix.

This orthogonality property is an important constraint on candidate interpolation methods for creating multiple strands from guide hairs. Any interpolation method which does not directly preserve this condition will not produce acceptable results.

Because the space curve is the integral of $\hat{T}$ over the path length, the computation of the space curve can be expressed in terms of the integral

$$R(s) = \int_0^s ds' \; U(s') \; , \qquad (15)$$

The solution for the space curve is then

$$\vec{x}(s) = \hat{e}_1 \cdot R(s) \cdot \mathbf{F}(0) \; , \qquad (16)$$

where $\hat{e}_1$ is a vector that extract the first element of the any other vector.

Precomputing $R(s)$ has a significant up-front cost in time and storage, but gives a fast way of constructing many identically-shaped strands differing only in position and orientation. Each additional strand requires just a few inner products at each vertex.

Since the application to hair rendering requires that strand shapes be interpolated, a process that may not preserve the orthogonality of $U(s)$, $U(s)$ or $R(s)$ are not used in the numerical implementation. Instead, the Frenet-Serret apparatus is calculated and applied directly in small steps along the curve.

3.2 Special Case: Constant Helicity

There is a special case in which the equations for the Frenet-Serret apparatus can be solved exactly. This is the case in which the helicity h is constant over the length of the strand. The equations for the Frenet-Serret basis can be solved exactly in this situation, for arbitrary curvature. The key to accomplishing this exact solution is a dimensional variable $q(s)$ defined as

$$q(s) = \int_0^s ds' \, \kappa(s') \; . \tag{17}$$

Because κ is a positive quantity, q is a monotonic increasing function of s. We can then change independent variables to q, and find the equations:

$$\frac{d\hat{T}(q)}{dq} = \hat{N}(q) \tag{18}$$

$$\frac{d\hat{N}(q)}{dq} = -\hat{T}(q) + h\hat{B}(q) \tag{19}$$

$$\frac{d\hat{B}(q)}{dq} = -h\hat{N}(q) \tag{20}$$

These are three couples equations in constant coefficients. Their solution is ($\ell \equiv q\sqrt{1+h^2}$).

$$\hat{T}(q) = \hat{T}(0) + \hat{N}(0)\frac{\sin\ell}{\sqrt{1+h^2}} + \frac{1-\cos\ell}{\sqrt{1+h^2}}\hat{\Gamma} \tag{21}$$

$$\hat{N}(q) = \hat{N}(0)\cos\ell + \hat{\Gamma}\sin\ell \tag{22}$$

$$\hat{B}(q) = \hat{B}(0) - \frac{h}{\sqrt{1+h^2}}\hat{N}(0)\sin\ell - \frac{h(1-\cos\ell)}{\sqrt{1+h^2}}\hat{\Gamma} \tag{23}$$

where

$$\hat{\Gamma} = \left(\frac{h}{\sqrt{1+h^2}}\hat{B}(0) - \frac{1}{\sqrt{1+h^2}}\hat{T}(0)\right) \; . \tag{24}$$

Although this is a ”special case”, there is a large range of problems which fall under this constraint or are close to it. A simple picture of what is going on is found in the model of a spring that is long and flexible. The curvature controls the cross-sectional shape of the spring. The helicity controls the tightness of the windings of the spring coils. Unfortunately, as discussed below, constant helicity prevents the coiled string from bending.

We can use this solution to examine the meaning of the helicity h and what the curve looks like in some limits. As $h \to 0$, this solution takes the form

$$\begin{aligned}
\hat{T}(q) &\to \hat{T}(0)\cos\ell + \hat{N}(0)\sin\ell + O(h) \\
\hat{N}(q) &\to -\hat{T}(0)\sin\ell + \hat{N}(0)\cos\ell + O(h) \\
\hat{B}(q) &\to \hat{B}(0) + O(h) \\
\ell(s) &\to \int_0^s ds'\kappa(s') + O(h)
\end{aligned}$$

so that the curve is located entirely within a plane when there is no helicity. Now ℓ has the interpretation as a winding number, in that the curve closes on itself each time ℓ exceeds a multiple of 2π.

In the other extreme, as $h \to \pm\infty$, the solution becomes

$$\begin{aligned}
\hat{T}(q) &\to \hat{T}(0) + O\left(\frac{1}{h}\right) \\
\hat{N}(q) &\to \pm\hat{B}(0)\sin\ell + \hat{N}(0)\cos\ell + O\left(\frac{1}{h}\right) \\
\hat{B}(q) &\to \hat{B}(0)\cos\ell \mp \hat{N}(0)\sin\ell + O\left(\frac{1}{h}\right) \\
\ell(s) &\to \int_0^s ds'|\tau(s')| + O\left(\frac{1}{h}\right)
\end{aligned}$$

which is a spring that has many windings of coil, and is approximately straight on the spatial scale of the windings, but curves once the arclength is large enough that $q(s) \sim |h|$.

However there is an important limitation of constant helicity curves: The tangent is always at a fixed angle to the axis of helicity. This is *Lancret's Theorem*, and is easily proven by noting that

$$\hat{T}(q) \cdot \left(\hat{N}(0) \times \hat{\Gamma}\right) = \frac{h}{\sqrt{1+h^2}} \tag{25}$$

is a constant. Thus the spring-like shape of curves with constant helicity cannot bend. Most problems of interest require spatial and temporal variations of helicity.

4 Numerical Implementation of Space Curves

The numerical implementation allows for going back and forth from parametrized position and the Frenet-Serret apparatus:

1. Given a starting position and orientation, as well as parametrized arc length, curvature, and torsion, we can synthesize the space curve position as a function of arc length.

2. Conversely, we can analyze the starting position and the derivatives of a space curve with respect to some arbitrary parameter into the starting orientation and parametrized versions of arc length, curvature, and torsion.

The next two subsections discuss these algorithms.

4.1 Space Curve Synthesis from Frenet-Serret Apparatus

We can avoid the cumbersome ordered-exponential needed for large leaps in arc length by repeatedly taking small steps, connecting nodes with helix segments. In this limit of constant curvature and torsion, $\mathbf{Q}$ commutes with itself and the ordered-exponential integral is equal to a simple matrix exponential.

The solution to equation 10 for small arc lengths over which the curvature and torsion do not vary is

$$\begin{aligned}\mathbf{F}(s+\Delta s) &= \left(\exp\left\{\int_s^{s+\Delta s} ds'\,\mathbf{Q}(s')\right\}\right)_+ \cdot\ \mathbf{F}(s) \\ &\approx \exp\left[\mathbf{Q}(s)\Delta s\right]\ \cdot\ \mathbf{F}(s)\end{aligned} \tag{26}$$

It turns out that there is a closed-form expression for $\exp\left(\mathbf{Q}(s)\Delta s\right)$:

$$\exp\begin{pmatrix} 0 & \kappa\Delta s & 0 \\ -\kappa\Delta s & 0 & \tau\Delta s \\ 0 & -\tau\Delta s & 0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ -b & d & e \\ c & -e & f \end{pmatrix} \tag{27}$$

and

$$a = 1 - \left[1-\cos\left(\Delta s\sqrt{\kappa^2+\tau^2}\right)\right]\left(\frac{\kappa^2}{\kappa^2+\tau^2}\right) \tag{28}$$

$$b = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\sin\left(\Delta s\sqrt{\kappa^2+\tau^2}\right) \tag{29}$$

$$c = \left[1-\cos\left(\Delta s\sqrt{\kappa^2+\tau^2}\right)\right]\left(\frac{\kappa\tau}{\kappa^2+\tau^2}\right) \tag{30}$$

$$d = \cos\left(\Delta s\sqrt{\kappa^2+\tau^2}\right) \tag{31}$$

$$e = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin\left(\Delta s\sqrt{\kappa^2+\tau^2}\right) \tag{32}$$

$$f = 1 - \left(1-\cos\left(\Delta s\sqrt{\kappa^2+\tau^2}\right)\right)\left(\frac{\kappa^2}{\kappa^2+\tau^2}\right) \tag{33}$$

The form of $\exp\left[\mathbf{Q}(s)\Delta s\right]$ is particularly simple if κ or τ is zero, reducing to a simple rotation matrix:

$$\exp\begin{pmatrix} 0 & \kappa\Delta s & 0 \\ -\kappa\Delta s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\kappa\Delta s & \sin\kappa\Delta s & 0 \\ -\sin\kappa\Delta s & \cos\kappa\Delta s & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{34}$$

$$\exp\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau\Delta s \\ 0 & -\tau\Delta s & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\tau\Delta s & \sin\tau\Delta s \\ 0 & -\sin\tau\Delta s & \cos\tau\Delta s \end{pmatrix} \tag{35}$$

Starting from $\mathbf{F}(0)$, we repeatedly multiply by $\exp\left(\mathbf{Q}\Delta s\right)$ to obtain $\mathbf{F}(s)$ all along the length of the curve. At the same time, we accumulate the position $\vec{x}(s)$ using equation 9:

$$\vec{x}\,(s+\Delta s) = \vec{x}\,(s) + \int_s^{s+\Delta s} \hat{T}(s')\,ds' \tag{36}$$

The integral above can be well approximated when curvature varies slowly over the arc length Δs as

$$\int_s^{s+\Delta s}\hat{T}(u)\,du \approx \frac{\hat{T}\,(s)+\hat{T}\,(s+\Delta s)}{2}\,\Delta s\,\mathrm{sinc}\left(\frac{\kappa\,(s)+\kappa\,(s+\Delta s)}{4}\,\Delta s\right) \tag{37}$$

This approximation is preferable to the simpler approximation $\hat{T}(s)\ \Delta s$ because it preserves arc length for constant curvature, and such shapes as a circle or helix are perfectly reproduced even with coarse sampling.

4.2 Space Curve Analysis into Frenet-Serret Apparatus

We assume that position and its derivatives with respect to some arbitrary parameter are available. In particular, nurbs curves are continuous and derivatives are available without taking finite differences.

The set of equations needed to get the arc length, curvature, and torsion are

$$s(r) = \int_{r_0}^{r} du \left|\frac{d\vec{x}(u)}{du}\right| \tag{38}$$

$$\hat{T}(s) = \left|\frac{d\vec{x}(r)}{dr}\right|^{-1} \frac{d\vec{x}(r)}{dr} \tag{39}$$

$$\kappa(s)\ \hat{N}(s) = \left|\frac{d\vec{x}(r)}{dr}\right|^{-2} \left(1 - \hat{T}(s)\hat{T}(s)\right) \cdot \frac{d^2\vec{x}(r)}{dr^2} \tag{40}$$

$$\kappa(s)\ \tau(s)\ \hat{B}(s) = \left|\frac{d\vec{x}(r)}{dr}\right|^{-3} \left(1 - \hat{T}(s)\hat{T}(s) - \hat{N}(s)\hat{N}(s)\right) \cdot \frac{d^3\vec{x}(r)}{dr^3} \tag{41}$$

When done quasi-continuously, κ and τ can be taken from the magnitudes of the above vectors, where the sign of κ is most conveniently taken to preserve the direction of $\hat{N}$ through inflection points and across straight line-segments, and the sign of τ is taken to preserve the right-handedness of $\{\hat{T}, \hat{N}, \hat{B}\}$.

In the discrete case, however, average values of κ and τ should be taken over the sampling interval Δs. These are best obtained not from the magnitudes above, but rather by comparing successive Frenet-Serret bases $\mathbf{F}\,(s)$ and $\mathbf{F}\,(s + \Delta s)$. Since this method effectively calculates the third derivative implicitly by finite difference, an explicit value is no longer needed as input.

Use normalized versions of equations 39, 40, and 41 to calculate $\hat{T}(0)$, $\hat{N}(0)$, and $\hat{B}(0)$ and save as initial Frenet-Serret basis.

Proceed down the strand in small steps Δr, saving at each step the previous results:

$$s + \Delta s \approx s\,(r) + \left|\frac{d\vec{x}(r)}{dr}\right|\ \Delta r \tag{42}$$

$$\hat{T}(s + \Delta s) = \left|\frac{d\vec{x}(r)}{dr}\right|^{-1} \frac{d\vec{x}(r)}{dr} \tag{43}$$

$$\hat{N}(s + \Delta s) = \left|\left(1 - \hat{T}(s)\hat{T}(s)\right) \cdot \frac{d^2\vec{x}(r)}{dr^2}\right|^{-1} \left(1 - \hat{T}(s)\hat{T}(s)\right) \cdot \frac{d^2\vec{x}(r)}{dr^2} \tag{44}$$

$$\hat{B}(s + \Delta s) = \hat{T}(s + \Delta s) \times \hat{N}(s + \Delta s) \tag{45}$$

First, $\hat{N}(s+\Delta s)$ and $\hat{B}(s+\Delta s)$ are negated if $\hat{B}(s+\Delta s)\cdot\hat{B}(s) < 0$. This avoids wild changes of orientation when the interval contains an inflection point or straight line segment.

Then, taking an outer product with $\mathbf{F}^T(s)$ in equation 26 and noting that $\mathbf{F}(s)$ is orthogonal gives

$$\begin{pmatrix} a & b & c \\ -b & d & e \\ c & -e & f \end{pmatrix} = \begin{pmatrix} \hat{T}(s+\Delta s)\cdot\hat{T}(s) & \hat{T}(s+\Delta s)\cdot\hat{N}(s) & \hat{T}(s+\Delta s)\cdot\hat{B}(s) \\ \hat{N}(s+\Delta s)\cdot\hat{T}(s) & \hat{N}(s+\Delta s)\cdot\hat{N}(s) & \hat{N}(s+\Delta s)\cdot\hat{B}(s) \\ \hat{B}(s+\Delta s)\cdot\hat{T}(s) & \hat{B}(s+\Delta s)\cdot\hat{N}(s) & \hat{B}(s+\Delta s)\cdot\hat{B}(s) \end{pmatrix}$$

Define $\arg(x,y)$ so that $\{x,y\} = \sqrt{x^2+y^2}\{\cos\arg(x,y), \sin\arg(x,y)\}$ for all real x and y and let

$$\rho = \frac{1}{\Delta s}\ \arg\left(d, \sqrt{b^2+e^2}\right) \tag{46}$$

Then,

$$\kappa(s) = \begin{cases} b & \rho^2 < 10^{-10} \\ \rho\cos\arg(b,e) & d \geq 0 \\ \operatorname{sign}(b)\sqrt{\rho^2\frac{1-a}{1-d}} & \text{otherwise} \end{cases} \tag{47}$$

$$\tau(s) = \begin{cases} e & \rho^2 < 10^{-10} \\ \rho\sin\arg(b,e) & d \geq 0 \\ \operatorname{sign}(e)\sqrt{\rho^2\frac{1-f}{1-d}} & \text{otherwise} \end{cases} \tag{48}$$

Figure 1 shows an example of a space curve from a numerical implementation of the approach in this section.

4.3 Creating New Strands from Guide Strands

The particular application of interest here is the construction of new hairs from *Guide Strands*. We take the collection of N space curves with associated initial position, orientation, and curvature and torsion profiles $\{\vec{x}_k(0), \mathbf{F}_k(0), \kappa_k(s), \tau_k(s)\}$ to characterize the guide strands.

Weights can be built to interpolate the properties of the guide hairs to generate an interpolated strand at any root position $\vec{x}(0)$. The weights must satisfy

$$\begin{aligned} w_k &\geq 0 \\ \sum_{k=1}^{N} w_k &= 1 \end{aligned}$$

For example, the choice $w_k \propto |\vec{x}(0)-\vec{x}_k(0)|^{-a}\ \left(\hat{S}\cdot\hat{T}_k(0)\right)\ \Theta\left(\hat{S}\cdot\hat{T}_k(0)\right)$ for $a \geq 0$, where $\Theta(x)$ is the Heaviside step function and $\hat{S}$ is the surface normal at $\vec{x}(0)$, is a possible choice of weights, with proportionality fixed by the normalization constraint in equation 49. The dot product with the local surface normal deweights guide hairs pointing away from the surface and ensures that guide hairs that are on the other side of a two-sided thin surface do not contribute at all.

What remains to be determined, however, is exactly what object(s) should be interpolated. There are three natural choices:

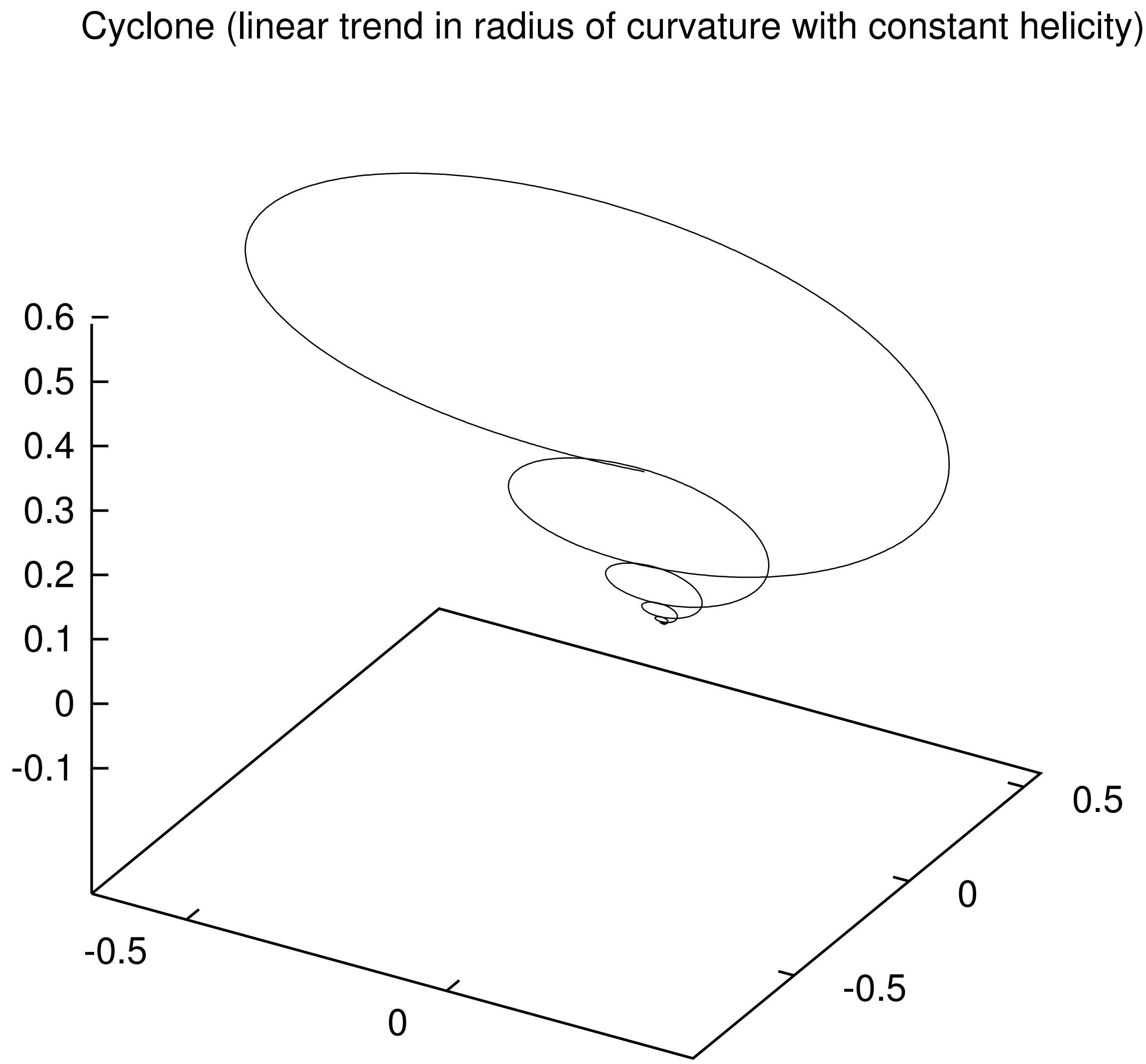

Figure 1: Example space curve using the Frenet-Serret Apparatus.

1. Interpolate the initial orientation and curvature and torsion profiles

$$\begin{aligned}
\tilde{\hat{T}}(s) &= \frac{\sum_{k=1}^{N} w_k \; \hat{T}_k(s)}{\left|\sum_{k=1}^{N} w_k \; \hat{T}_k(s)\right|} \\
\tilde{\kappa}(s)\; \tilde{\hat{N}}(s) &= \frac{\sum_{k=1}^{N} w_k \; \kappa_k(s) \; \left(1 - \tilde{\hat{T}}(s)\tilde{\hat{T}}(s)\right) \cdot \hat{N}_k(s)}{\left|\sum_{k=1}^{N} w_k \; \left(1 - \tilde{\hat{T}}(s)\tilde{\hat{T}}(s)\right) \cdot \hat{N}_k(s)\right|} \\
\tilde{\kappa}(s)\; \tilde{\tau}(s)\; \tilde{\hat{B}}(s) &= \frac{\sum_{k=1}^{N} w_k \; \kappa_k(s)\; \tau_k(s) \; \left(1 - \tilde{\hat{T}}(s)\tilde{\hat{T}}(s) - \tilde{\hat{N}}(s)\tilde{\hat{N}}(s)\right) \cdot \hat{B}_k(s)}{\left|\sum_{k=1}^{N} w_k \; \left(1 - \tilde{\hat{T}}(s)\tilde{\hat{T}}(s) - \tilde{\hat{N}}(s)\tilde{\hat{N}}(s)\right) \cdot \hat{B}_k(s)\right|}
\end{aligned}$$

   and then synthesize the final strand profile from these as described previously.

2. Construct the $U_k(s)$ matrix for each guide strand, and interpolate the elements of those matrices. The difficulty with this approach is that each of the $U_k(s)$ are orthognal matrices, whereas a simple interpolation $\sum_k w_k U_k(s)$ is not orthogonal. This interpolation must be modified to preserve orthogonality, and approached with caution.

3. Construct the $R_k(s)$ matrix for each guide strand, and interpolate the elements of those matrices. There are caveats to this approach as well, for reasons related to the problems with interpolating the U matrices. An interpolated R matrix may cause the actual length of the strand from the root to a point s to not be equal to the arclength. This error would cascade problems into other geometric and rendering issues. A possible way to overcome that is to compute the actual length of the interpolated strand, and rescale the arclength to match.

In light of the complications that interpolation induces in options (2) and (3) above, the present plan is to implement option (1) as the safest. If appropriate constrained-interpolation methods are found, options (2) or (3) may yet prove to be more efficient and as accurate.

5 Conclusions

The Frenet-Serret Apparatus is an excellent method of describing space curves in our application. It neatly separates the intrinsic shape of the curve from spatial orientation. Numerical implementation of the necessary code is straightforward. By using curvature and torsion as the underlying variables, we are assured that the strands will maintain fixed length no matter what the shape. In fact, curvature and torsion are unconstrained variables which preserve the quality of the space curve representation.