3D Cloud Scene Simulator V2
Algorithm for Scattering

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Contents

1 Introduction

The purpose of this document is to describe the algorithm(s) used in the Areté 3D Cloud Scene Simulator V2, developed by Areté Associates under support by ONR in the Infrared Analysis, Measurements, and Modeling Program (IRAMMP). The actual implementation in code is discussed in a separate document, '"Cloud Simulator 2.1"'.

As radiation propagates through clouds it is scattered and absorbed by the liquid water droplets and ice crystals. In the infrared, single scatter albedos are on the order of 0.8, and in the visible they can exceed 0.99. At these magnitudes, radiation will scatter many times before it is absorbed, and any image of a cloud must necessarily include a signficant fraction of multiply-scattered radiation. Parcels of cloud also emit thermal radiation isotropically as a blackbody at the local temperature (local thermodynamic equilibrium), and the sum total distribution of thermal radiation from all of the parcels is believed to be relatively unchanged by scattering. Thus, the effects of scattering on thermal emission has generally been assumed to be small and scattering has been ignored for the purposes of studying the mean structure of the clouds as they appear in infrared images. This has justified the study of clouds in longwave bands using models that ignore scattering or characterized it in a simple parametric way. However, it is expected that scattering must be included in any model or simulation of clouds in any visible or infrared bands if one wants a detailed understanding of the "edge" region or of the fluctuations seen within the cloud.

For wavelengths in the midwave region and smaller, solar radiation becomes an increasingly important part of the observed cloud radiance. Solar radiation incident on a cloud system is highly collimated and scattering greatly changes the angular distribution of the sunlight as it propagates through the cloud. In this case, a reliable calculation of scattering is essential in obtaining a good representation of the cloud image.

The Infrared Analysis, Measurements, and Modeling Program (IRAMMP) has an interest in developing an accurate, physically-based cloud scene simulator which can operate at all infrared bands. Clouds are an important souce of clutter for infrared imaging systems that are both ship-based and airborne. In the airborne scenario, clouds provide the background against which other airborne vehicles must be distinguished. For ship-based scenarios, clouds can generate inhomogeneous ocean surface clutter through their ability to cast shadows and emit solar and thermal light. Within IRAMMP goals, the 3D Cloud Scene Simulator described in the following sections is undergoing implementation as a module in the IRTool workstation software.

Section 2 contains a description of the general framework for the 3D Cloud Scene Simulator, including cloud/sensor geometry. The raytracing paradigm is introduced as an efficient method of systematically implementing any cloud scene simulator, whether scattering is present or not. In section 3, this framework is used to evaluate the important first cases of no- and single-scattering events. These provide the energy content to fuel multiple scattering, as discussed in section 4, where the approximate multiple scattering algorithm is presented. Section 5 contains a description of the specific use of this algorithm when the sun and sky radiance are acting as external sources and section 6 summarizes the algorithm.

2 Radiative Transfer Formulated as Raytracing

A natural way to describe the propagation of radiation is via raytracing. This section is devoted to formulating the exact raytracing equivalent to the radiative transfer problem. Unfortunately, the implementation of an exact raytrace scheme when a large number of scatterings occur is not computationally feasible since the number of rays needed grows with each scattering. This report describes an approximate raytrace algorithm that efficiently deals with large numbers of scattering events. In this algorithm, only certain dominant paths are traced. All other paths are dealt with by an analytical integration with respect to these dominant paths.

When a camera images a cloud, the detected radiance is the sum of contributions from the cloud and atmosphere:

$$L(\vec{x}_c, \hat{n}) = L_{\text{path}} + T_{\text{path}} L_{\text{cloud}}(\vec{x}_c, \hat{n}) \ . \tag{1}$$

The terms of the equation are:

$\vec{x}_c$	Position of camera.
$\hat{n}$	Look direction for a particular pixel.
L_{path}	The in-band atmospheric radiance occuring between the cloud(s) and camera.
T_{path}	Atmospheric transmittance between cloud(s) and camera.
L_{cloud}	Radiance emitted by the cloud(s)

The cloud that is imaged is a three dimensional volume containing spatially varying distributions of temperature (for the thermal emissions) and liquid/ice water density. For algorithmic purposes, the cloud or cloud system is contained within a rectangular box of arbitrary size, but the cloud need not fill it. The box serves as a control surface on which to manage raytracing information.

The cloud radiance is determined by thermal emission from the cloud itself, as well as sunshine and earthshine scattering through the cloud. In order to calculate L_{cloud} , we need to propagate all external and internal sources of radiation through the cloud to the point on the cloud box surface that the camera is looking at. To do this, we base our algorithms on an enhanced raytracing scheme that is derived directly from the radiative transfer equation. This allows us to write a core raytrace code which can be applied no matter what level of sophistication is applied to the modeling of the cloud physics.

We begin by casting the solution of the radiative transfer equation in the form of multiple raytraces. We want to include in this formulation the impact of the sun and earthshine, and conceptually other sources as well. For now all of the external sources are lumped together as a radiance $L_{\text{source}}(\vec{x}, \hat{n})$. In section 5, the specific cases of collimated solar light and diffuse earthshine are treated.

The radiative transfer equation for the radiance at any position in the volume is

$$\{\hat{n} \cdot \nabla + c(\vec{x})\} \, L_{\text{cloud}}(\vec{x}, \hat{n}) = b(\vec{x}) \int d\Omega' \; P(\hat{n} \cdot \hat{n}') \; L_{\text{cloud}}(\vec{x}, \hat{n}') \; + \; a(\vec{x}) B(\vec{x}) \ . \tag{2}$$

$B(\vec{x})$ is the isotropic blackbody radiance emitted by point $\vec{x}$, $a(\vec{x})$ is the spatially varying absorption coefficient, $b(\vec{x})$ is the spatially varying scattering coefficient, $P(\hat{n} \cdot \hat{n}')$ is the spatially invariant phase function, and $c = a + b$. The decomposition of the scattering term of the equation in terms of a spatially varying scattering magnitude $b(\vec{x})$ and a spatially invariant phase function $P(\hat{n} \cdot \hat{n}')$ is appropriate for cloud conditions in which the relative size distribution and composition of water/ice droplets is uniform throughout the cloud, but with spatially varying density.

In order to show the dependence of L_{cloud} on the external source radiance, and guarantee that boundary conditions at the cloud edges are satisfied, we decompose the radiance into the unscattered external source component plus a component for all of the thermal emission and multiple scattering. While this approach is perfectly general and is the best way to handle collimated sources such as the sun, when dealing with diffuse sources such as earthshine, we will see that this decomposition is not the most efficient.

The external source radiance is attenuated and scattered as it penetrates the cloud volume. Within the cloud, the unscattered portion of the attenuated source satisfies

$$\{\hat{n} \cdot \nabla + c(\vec{x})\} \, L_{\text{source}}^{\text{unscattered}}(\vec{x}, \hat{n}) = 0 \ , \tag{3}$$

which has the solution

$$L_{\text{source}}^{\text{unscattered}}(\vec{x}, \hat{n}) = L_{\text{source}}^{\infty}(\vec{x}, \hat{n}) \ \exp\left\{-\int_0^{\infty} ds \ c(\vec{x} - \hat{n}s)\right\} \ . \tag{4}$$

The quantity $L_{\text{source}}^{\infty}(\vec{x}, \hat{n})$ is the source radiance in the absence of the cloud.

Decomposing the total radiance in terms of the unscattered external source and all other radiance as

$$L_{\text{cloud}}(\vec{x}, \hat{n}) = L_{\text{source}}^{\text{unscattered}}(\vec{x}, \hat{n}) + \delta L(\vec{x}, \hat{n}) \ , \tag{5}$$

and using equation 2 implies that δL satisfies

$$\begin{aligned} \{\hat{n} \cdot \nabla + c(\vec{x})\} \, \delta L(\vec{x}, \hat{n}) \ &= \ b(\vec{x}) \int d\Omega' \ P(\hat{n} \cdot \hat{n}') \ \delta L(\vec{x}, \hat{n}') \\ &+ \ a(\vec{x}) B(\vec{x}) + b(\vec{x}) B_{\text{source}}(\vec{x}, \hat{n}) \ , \end{aligned} \tag{6}$$

where

$$B_{\text{source}}(\vec{x}, \hat{n}) = \int d\Omega' \ P(\hat{n} \cdot \hat{n}') \ L_{\text{source}}^{\text{unscattered}}(\vec{x}, \hat{n}') \ . \tag{7}$$

The connection to raytracing is made by noting that equation 6 can be rewritten in a Global Formulation

$$\delta L(\vec{x}, \hat{n}) = \int_0^{\infty} ds \ \int d^3x' \ d\Omega' \ G(s, \vec{x}, \hat{n}; \vec{x}', \hat{n}') \ \{a(\vec{x}') B(\vec{x}') + b(\vec{x}') B_{\text{source}}(\vec{x}', \hat{n}')\} \ , \tag{8}$$

where the Evolution Operator G satisfies the time-dependent radiative transfer equation

$$\left\{\frac{\partial}{\partial s} + \hat{n} \cdot \nabla + c(\vec{x})\right\} G(s, \vec{x}, \hat{n}; \vec{x}', \hat{n}') \ = \ b(\vec{x}) \int d\Omega'' \ P(\hat{n} \cdot \hat{n}'') \ G(s, \vec{x}, \hat{n}''; \vec{x}', \hat{n}') \ , \tag{9}$$

with the initial condition

$$G(s = 0, \vec{x}, \hat{n}; \vec{x}', \hat{n}') = \delta\left(\vec{x} - \vec{x}'\right) \ \delta\left(\hat{n} - \hat{n}'\right) \ . \tag{10}$$

G represents the angular distribution and density of rays a distance s from a unit source located at $\vec{x}'$ and emitting in direction $\hat{n}'$.

We generate algorithms for modeling radiative transfer in the cloud simulator by developing various levels of approximations for G. Each approximation is equivalent to a raytrace scheme. This provides an underlying structure for the code independent of the physical approximations, and simplifies the implementation of any algorithm, as well as the incorporation of eventual improvements.

3 No Scatter and Single Scatter Limits

This section develops raytrace algorithms corresponding to the no and single scatter approximations. The development of these algorithms is useful for both understanding how the global formulation of the previous section can be used to generate raytrace algorithms and for deriving some results that will be used in the next section. We begin by showing explicitly that the existing absorption only raytrace algorithm is derived in the limit of no scattering.

When the scattering coefficient (b) is zero, the exact solution for G is

$$G_{\text{no scatter}}(s,\vec{x},\hat{n};\vec{x}',\hat{n}') = \delta\left(\vec{x}-\hat{n}s-\vec{x}'\right)\ \delta\left(\hat{n}-\hat{n}'\right)\ \exp\left\{-\int_0^s ds'\ c(\vec{x}-\hat{n}(s-s'))\right\}. \qquad (11)$$

Even though $c = a$ in this case, we have kept the total extinction coefficient in this expression for use below. For purely thermal emission, $L_{\text{source}} = 0$ and we recover the standard absorption only algorithm

$$L_{\text{cloud}}(\vec{x},\hat{n}) = \int_0^\infty ds\ B(\vec{x}-\hat{n}s)\ a(\vec{x}-\hat{n}s)\ \exp\left\{-\int_0^s ds'\ a(\vec{x}-\hat{n}(s-s'))\right\}. \qquad (12)$$

In this algorithm, rays simply travel in a straight line from the source and are attenuated by the absorption coefficient.

Since the source term already represents a single scatter event for the external radiance, we can use $G_{\text{no scatter}}$ to describe the case of single scattering of the external radiance, with no additional scattering events. In this case, the cloud radiance is

$$\begin{aligned}\delta L_(\vec{x},\hat{n}) = \int_0^\infty ds \quad & \{B(\vec{x}-\hat{n}s)\ a(\vec{x}-\hat{n}s) + B_{\text{source}}(\vec{x}-\hat{n}s,\hat{n}s)b(\vec{x}-\hat{n}s)\} \\ & \times \exp\{-\textstyle\int_0^s ds'\ a(\vec{x}-\hat{n}(s-s'))\}. \end{aligned} \qquad (13)$$

In this algorithm, the external radiance travels in a straightline, scatters once, and then travels in another straightline. This is a suitable first algorithm to be implemented, since it contains the primary scattering event for the external source. It will be valid in situations where subsequent scatterings do not significantly alter the angular distribution of the traveling photons.

A perturbation expansion can be used to add in the effect of additional scatterings. Here, we use it to incorporate single scattering of the thermal distribution. The zeroth order solution is $G_{\text{no scatter}}$ above, and the first order solution is

$$\begin{aligned} G_{\text{single scatter}}(s,\vec{x},\hat{n};\vec{x}',\hat{n}) &= G_{\text{no scatter}}(s,\vec{x},\hat{n};\vec{x}',\hat{n}') \\ &+ \int_0^s ds' \int d^3x''\ d\Omega''\ G_{\text{no scatter}}(s-s',\vec{x},\hat{n};\vec{x}'',\hat{n}'') \\ &\times\ b(\vec{x}'') \int d\Omega'''\ P(\hat{n}''\cdot\hat{n}''')G_{\text{no scatter}}(s',\vec{x}'',\hat{n}''';\vec{x}',\hat{n}') \end{aligned} \qquad (14)$$

Substituting the expression for $G_{\text{no scatter}}$, this is

$$\begin{aligned} G_{\text{single scatter}}(s,\vec{x},\hat{n};\vec{x}',\hat{n}') &= \delta\left(\vec{x}-\hat{n}s-\vec{x}'\right)\ \delta\left(\hat{n}-\hat{n}'\right)\ \exp\left\{-\int_0^s ds'\ c(\vec{x}-\hat{n}(s-s'))\right\} \\ &+ \int_0^s ds'\ \delta\left(\vec{x}-\hat{n}(s-s')-\hat{n}'s'-\vec{x}'\right) \\ &\times\ b(\vec{x}-\hat{n}(s-s')-\hat{n}'s')\ P(\hat{n}\cdot\hat{n}') \end{aligned}$$

$$\times \quad \exp\left\{-\int_0^{s-s'} ds''\ c(\vec{x}-\hat{n}(s-s'-s''))\right\}$$

$$\times \quad \exp\left\{-\int_0^{s'} ds''\ c(\vec{x}-\hat{n}(s-s')-\hat{n}'(s'-s''))\right\} \tag{15}$$

Using this expression, the cloud radiance is

$$\begin{aligned}
\delta L(\vec{x},\hat{n}) &= \int_0^\infty ds\ \{B(\vec{x}-\hat{n}s)a(\vec{x}-\hat{n}s) + B_{\text{source}}(\vec{x}-\hat{n}s,\hat{n})b(\vec{x}-\hat{n}s)\} \\
&\times \exp\left\{-\int_0^s ds'\ c(\vec{x}-\hat{n}(s-s'))\right\} \\
&+ \int_0^\infty ds \int_0^s ds' \int d\Omega'\ B(\vec{x}-\hat{n}(s-s')-\hat{n}'s')\ a(\vec{x}-\hat{n}(s-s')-\hat{n}'s') \\
&\times b(\vec{x}-\hat{n}(s-s')-\hat{n}'s')\ P(\hat{n}\cdot\hat{n}') \\
&\times \exp\left\{-\int_0^{s-s'} ds''\ c(\vec{x}-\hat{n}(s-s'-s''))\right\} \\
&\times \exp\left\{-\int_0^{s'} ds''\ c(\vec{x}-\hat{n}(s-s')-\hat{n}'(s'-s''))\right\}
\end{aligned} \tag{16}$$

Note that B_{source} is not in the second term because B_{source} is already a single scatter quantity. This algorithm completely describes the propagation of all radiation in the problem including up to single scattering.

On first glance, equation 16 appears to be impractical as a raytrace algorithm. The integral factor $\int d\Omega'$ indicates that at each point along a ray trace line, a group of additional daughter rays would be traced in all 4π directions. This would quickly mushroom the computational load unless the source is highly directional, or some other constraint limits the solid angle over which the integration must be performed. For solar light we will see that the source is proportional to the phase function. In cases such as this for which the phase function is highly forward peaked, we can limit the angular range of integration to a range believed to contain most of the contribution, making this approach computationally viable. Note that if we were to try and extend this approach to achieve multiple scattering, the Nth scatter event would have N solid angle integrations of the phase function, and the minimum angular range we can restrict the evaluation to would become larger and larger.

We will see in section 4 that when multiple scattering is incorporated, the result requires a computational load similar to or less than equation 16.

4 Multiple Scatter Algorithms

For multiple scattering conditions, we can adopt a WKB approximation similar to that used in several other works[1, 2]. However, the spatially variability of the optical properties in the cloud problem is a new wrinkle on the WKB approach which has not be previously attacked. We will outline below the derivation of the expression for G, including spatial variability of the optical properties. In order to understand it fully, the two references are also needed to fill in gaps in the intermediate steps math.

Including spatial variability in a and b, the path integral expression for G is

$$G(s,\vec{x},\hat{n};\vec{x}',\hat{n}') = G_{\text{no scatter}}(s,\vec{x},\hat{n};\vec{x}',\hat{n}')$$

$$
\begin{aligned}
+ & \int \left[d\hat{\beta}\right] [dp] \; \delta\left(\vec{x} - \vec{x}' - \int_0^s ds' \; \hat{\beta}(s')\right) \; \delta\left(\hat{\beta}(0) - \hat{n}'\right) \; \delta\left(\hat{\beta}(s) - \hat{n}\right) \\
\times & \; \exp\left\{ - \int_0^s ds' \; c\left(\vec{x}' + \int_0^{s'} ds'' \; \hat{\beta}(s'')\right)\right\} \\
\times & \; \exp\left\{ i \int_0^s ds' \; \dot{\hat{\beta}}(s') \cdot \vec{p}(s') \right\} \\
\times & \; \left[\exp\left\{ \int_0^s ds' \; b\left(\vec{x}' + \int_0^{s'} ds'' \; \hat{\beta}(s'')\right) Z(\vec{p}(s'))\right\} - 1 \right] . \qquad (17)
\end{aligned}
$$

In this expression, Z is the Fourier transform of the phase function and $\vec{p}$ is the Fourier transform variable. The phase function is normalized to one implying $Z(0) = 1$. $\hat{\beta}(s')$ is the direction that the ray originating at $\vec{x}'$ is traveling at time s'. The displacement of a ray relative to $\vec{x}'$ at time s is given by

$$
\Delta\vec{x}'(s, \hat{\beta}) = \int_0^s ds' \hat{\beta}(s'). \qquad (18)
$$

The path integral over $\hat{\beta}$ is equivalent to integrating over all possible paths that a ray can take from $\vec{x}'$. The delta functions enforce the constraint that the ray starts moving in direction $\hat{n}'$ and ends at point $\vec{x}$ moving in direction $\hat{n}$.

4.1 WKB Algorithm for Forward-Peaked Phase Function

Because of the highly forward peaked character of the phase function, we will be interested in capturing mostly just that forward part. A simple expression used frequently is

$$
P_{FP}(\Theta, \mu) = \frac{1}{2\pi\mu} \exp\left\{ -\frac{\Theta^2}{2\mu} \right\} . \qquad (19)
$$

which corresponds to

$$
Z(\vec{p}) = \exp\left\{ -\frac{\mu}{2} p^2 \right\} . \qquad (20)
$$

The width parameter μ is relatively small (e.g. $\mu \approx 0.035$ in the ocean and $\mu \approx 0.1$ in clouds) corresponding to a strongly forward peaked phase function. This phase function differs from a realistic phase function in that it ignores the backscattered part. In several appropriate places in the WKB evaluation of G, the smallness of μ is used to expand Z in just its first two terms of a Taylor expansion.

The spatial variability of the optical properties is handled by changing variables from s to the dimensionless parameter ℓ defined as

$$
\ell'(s', \vec{x}', \hat{\beta}) = \int_0^{s'} ds'' \; b\left(\vec{x}' + \int_0^{s''} ds''' \; \hat{\beta}(s''')\right) . \qquad (21)
$$

In a uniform system $\ell' = bs'$ and there is no difference between using either ℓ or s as a variable. In the non-uniform system the relationship between the two variables becomes non-trivial and ℓ is the natural variable to use because it roughly measures the number of times a ray scatters while moving along a given path.

In terms of ℓ , the relationship between $\Delta\vec{x}'$ and $\hat{\beta}$ is

$$
\Delta\vec{x}'(\ell', \vec{x}', \hat{\beta}) = \int_0^{\ell'} d\ell'' \frac{\hat{\beta}(\ell'')}{b(\vec{x}' + \Delta\vec{x}'(\ell'', \vec{x}', \hat{\beta}))}. \qquad (22)
$$

Here $\Delta\vec{x}'$ is determined implicitly by equation 22, as opposed to the explicit expression of equation 18. Nonetheless, since $\Delta\vec{x}'$ at a given ℓ only depends on the values of $\Delta\vec{x}'$ at earlier values of ℓ it is straightforward to develop a numerical scheme that solves equation 22 for successive values of ℓ. Once $\Delta\vec{x}'$ is determined, the scattering part of equation 17 becomes

$$\begin{aligned} G_{\text{scatter}}(s,\vec{x},\hat{n};\vec{x}',\hat{n}') &= \int \left[d\hat{\beta}\right][dp]\ \delta\left(\vec{x}-\vec{x}'-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta})\right)\ \delta\left(\hat{\beta}(0)-\hat{n}'\right)\ \delta\left(\hat{\beta}(s)-\hat{n}\right) \\ &\times\ \exp\left\{-\int_0^\ell d\ell'\ \frac{c(\vec{x}'+\Delta\vec{x}'(\ell',\vec{x}',\hat{\beta}))}{b(\vec{x}'+\Delta\vec{x}'(\ell',\vec{x}',\hat{\beta}))}\right\} \\ &\times\ \exp\left\{i\int_0^\ell d\ell'\ \frac{d\hat{\beta}(\ell')}{d\ell'}\cdot\vec{p}(\ell')\right\} \\ &\times\ \left[\exp\left\{\int_0^\ell d\ell'\ Z(\vec{p}(\ell'))\right\}-1\right]. \end{aligned} \tag{23}$$

Here we have used the shorthand notation $\ell \equiv \ell(s,\vec{x}',\hat{\beta})$.

The WKB approximation consists of approximating the path integral in equation 23 by expanding about the paths of least attenuation and keeping the quadratic fluctuations, which can then be integrated. In equation 23 the only explicit spatial dependence in the integrand comes from the ratio of c/b in the first exponential. If we assume that the variability of c/b does not significantly alter the path which contributes the most to the integral then the least attenuation path is simply determined by

$$\frac{\delta}{\delta\hat{\beta}}\int_0^\ell \left|\frac{d\hat{\beta}(\ell')}{d\ell'}\right|^2 = 0. \tag{24}$$

For each value of ℓ,$\hat{n}$, and $\hat{n}'$ the solution of eq. 24 consists of all $\hat{\beta}$ that uniformly rotate in the variable ℓ' from $\hat{n}'$ to $\hat{n}$. These solutions are labeled by the winding number n_w (which takes on all integer values) and are denoted

$$\hat{\beta}_0^{n_w}(\ell';\ell,\hat{n},\hat{n}'). \tag{25}$$

The specific solutions that rotate

$$\hat{n}' = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \tag{26}$$

to

$$\hat{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{27}$$

over a period ℓ are

$$\hat{\beta}_0^{n_w}(\ell';\ell,\hat{z},\hat{n}') = \begin{pmatrix} \sin\left(\theta(\ell')\right)\cos\left(\phi\right) \\ \sin\left(\theta(\ell')\right)\sin\left(\phi\right) \\ \cos\left(\theta(\ell')\right) \end{pmatrix} \tag{28}$$

where

$$\theta(\ell') = \theta + (2\pi n_w - \theta)\ell'/\ell. \tag{29}$$

The general solutions are constructed from the above solution by applying rotation matrices. Let $R(\hat{n})$ be the rotation matrix that rotates $\hat{z}$ to $\hat{n}$, then

$$\hat{\beta}_0^{n_w}(\ell';\ell,\hat{n},\hat{n}') = R(\hat{n})\hat{\beta}_0^{n_w}(\ell';\ell,\hat{z},R^{-1}(\hat{n})\hat{n}'). \tag{30}$$

Refering to equation 22, we see that these $\hat{\beta}$ generate paths in physical space with a local radius of curvature $\sim 1/b$. Hence, when b is small the paths are approximately straight lines and when b is large they are highly curved. The WKB approximation assumes that the actual paths traveled are small fluctuations on the base path generated by $\hat{\beta}_0^{n_w}$.

The WKB approximation can now be applied as in previous work. For a given value of ℓ the evolution operator is

$$\begin{aligned} G_{\text{scatter}}(\ell,\vec{x},\hat{n};\vec{x}',\hat{n}') \approx & \sum_{n_w} \exp\left\{-\int_0^\ell d\ell' \, \frac{c(\vec{x}'+\Delta\vec{x}'(\ell',\vec{x}',\hat{\beta}_0^{n_w}))}{b(\vec{x}'+\Delta\vec{x}'(\ell',\vec{x}',\hat{\beta}_0^{n_w}))}\right\} \\ & \times \left(e^\ell - 1\right) P_{\text{FP}}\left(\Theta + 2n_w\pi, \mu\ell/(1-\exp(-\ell)\right) \\ & \times \delta\left(\vec{x}-\vec{x}'-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta}_0^{n_w})\right) \ , \end{aligned} \tag{31}$$

where $\Theta = cos^{-1}(\hat{n}\cdot\hat{n}')$ is the angle between the incoming and outgoing directions. Observe that the effective width of the phase function grows with increasing ℓ, reflective of the fact that it becomes easier to scatter to a given angle if a large number of scatterings have occurred. The WKB approximation predicts that the average angle scattered through grows with ℓ like

$$\Delta\Theta_{\text{avg}} \sim \frac{\mu\ell}{1-\exp(-\ell)} \ . \tag{32}$$

Eq. 31 represents the WKB approximation for the evolution operator in a spatially varying system. The assumptions and approximations needed to arrive at this point are as follows:

1. The phase function is sharply forward peaked. Consequently, large angle backscattering is in principle poorly modeled here. This is not as serious a concern as it could be, because we are explicitly including the first scattering event as sources within the cloud (see the section below). The multiple scattering accounted for in equation 31 is for the multiple forward scatters after the first event. The forward-peaked assumption will be weakly removed in the next subsection.

2. The spatial variability of c/b does not alter the path of least attenuation from one point in the cloud to another. This allows us to handle an arbitrary realization of optical property fluctuations in a simple and robust way. The behavior of c/b in clouds is not well known. In many simple cloud models, however, both quantities are directly proportional to the liquid water content implying c/b is a constant and our formulation directly applies. If one is studying a cloud model with largely varying values of c/b then this assumption needs to be carefully examined.

3. Intrinsic spatially broadening of a point source is not accounted for in equation 31. There is spatial spreading due to the angular broadening of the rays and corresponding sampling of a larger volume of the cloud. This approximation was made because clouds are spatially extended over scales larger than the ignored intrinsic spreading, so that the angular spreading coupled to the extended spatial range of the cloud dominates.

The cloud radiance is calculated from the evolution operator using equation 8. This requires the calculation of the evolution operator for each value of s. For a given ℓ and path β the path length is

$$s_{\rm WKB}[\ell, \vec{x}', \hat{\beta}_0^{n_w}] = \int_0^\ell d\ell' \, \Delta\vec{x}' \left(\ell', \vec{x}', \hat{\beta}_0^{n_w}(\ell'; \ell, \hat{n}, \hat{n}')\right) \ . \tag{33}$$

Note that it is possible for more than one value of ℓ to correspond to a single combination of $s_{\rm WKB}$, n_w, $\hat{n}$, and $\hat{n}'$. The integral along ℓ is converted to an integral along $s_{\rm WKB}$ via

$$ds_{\rm WKB} = \left| \frac{ds_{\rm WKB}}{d\ell} \right| d\ell \tag{34}$$

yielding

$$\begin{aligned} \delta L(\vec{x}, \hat{n}) \ = \ & \int_0^\infty ds \{a(\vec{x} - \hat{n}s) B(\vec{x} - \hat{n}s) + b(\vec{x} - \hat{n}s) B_{\rm source}(\vec{x} - \hat{n}s, \hat{n})\} \exp\left\{ - \int_0^s ds' \ c(\vec{x} - \hat{n}(s - s')) \right\} \\ & + \ \sum_{n_w} \int d\Omega' \int_0^\infty d\ell \left| \frac{ds_{\rm WKB}[\ell, \vec{x}', \hat{\beta}_0^{n_w}]}{d\ell} \right| \\ & \times \ \left\{ a(\vec{x} - \Delta\vec{x}'(\ell, \vec{x}', \hat{\beta}_0^{n_w})) B(\vec{x} - \Delta\vec{x}'(\ell, \vec{x}', \hat{\beta}_0^{n_w})) + b(\vec{x} - \Delta\vec{x}'(\ell, \vec{x}', \hat{\beta}_0^{n_w})) B_{\rm source}(\vec{x} - \Delta\vec{x}'(\ell, \vec{x}', \hat{\beta}_0^{n_w} \right. \\ & \times \ \exp\left\{ - \int_0^\ell d\ell' \frac{c(\vec{x} - \Delta\vec{x}'(\ell, \vec{x}', \hat{\beta}_0^{n_w}) + \Delta\vec{x}'(\ell', \vec{x}', \hat{\beta}_0^{n_w}))}{b(\vec{x} - \Delta\vec{x}'(\ell, \vec{x}', \hat{\beta}_0^{n_w}) + \Delta\vec{x}'(\ell', \vec{x}', \hat{\beta}_0^{n_w}))} \right\} \\ & \times \ \left(e^\ell - 1 \right) P_{\rm FP} \left(\Theta + 2n_w\pi, \mu\ell/(1 - \exp(-\ell)) \right) \ . \end{aligned}$$

where we have kept the unscattered piece in terms of the variable s.

From a computational point of view, equation 35 is somewhat simpler to implement than the single scatter solution in equation 16. The unscattered first term in both approaches is identical. In the second term, both approaches launch groups of rays to handle the integration $\int d\Omega'$. The number of rays is controlled in both approaches by the width of the phase function (which in the multiple scattering case grows with ℓ). The single scatter approach then requires an additional integral $\int_0^s ds'$ to account in detail for each possible singly scattered path for each of the launched rays. The multiple scatter approach accounts for the scattering by using curved paths $\vec{x}' + \Delta\vec{x}'(\ell, \vec{x}', \hat{\beta}_0^{n_w})$, which do depend on the direction of each launched ray but can be computed at the ray burrows along without the additional integration.

4.2 WKB Algorithm for General Phase Functions

The previous WKB result has a fairly straightforward structure. Rays move along paths $\hat{\beta}_0^{n_w}$ that rotate at a uniform rate with respect to the evolving scattering length ℓ. In order to accomplish this the average scatter occurs thru an angle

$$(\Theta + 2n_w\pi)\sqrt{(1 - \exp(-\ell))/\ell}. \tag{36}$$

For large values of ℓ the ray needs to scatter less each scattering event to get to a given final angle and hence is increasingly sensitive to the forward part of the phasefunction compared to paths corresponding to smaller values of ℓ. Given this interpretation, it is possible to heuristically construct a generalized WKB solution for an arbitrary phase function $P(\Theta)$.

We define the multiple-scattering phase function $P_{\rm MS}$ in terms of P as

$$P_{\rm MS}(\Theta,\alpha_\ell,n_w) = \frac{1}{N(\alpha_\ell)} P\left([\Theta + 2n_w\pi]/\alpha_l\right), \tag{37}$$

where

$$\alpha_\ell = \sqrt{\frac{\ell}{1-\exp(-\ell)}} \tag{38}$$

and

$$N(\alpha_\ell) = 2\pi \int_0^{2\pi\alpha_\ell} d\Theta \; |\sin(\Theta)| P(\Theta/\alpha_l). \tag{39}$$

The multiple-scattered phase function is normalized so that

$$\sum_{n_w} \int d\Omega P_{\rm MS}(\Theta,\alpha_l,n_w) = 1 \tag{40}$$

and has the interpretation as being the probability for a ray to scatter thru an angle Θ after traversing ℓ scattering lengths. In the limit $\ell \to \infty$, $P_{\rm MS}$ becomes isotropic.

If we assume that the effect of small fluctuations around $\hat{\beta}_0^{n_w}$ is the same for P as for $P_{\rm FP}$ then we can write the WKB result for the arbitrary phase function as

$$\begin{aligned}
\delta L(\vec{x},\hat{n}) &= \int_0^\infty ds\{a(\vec{x}-\hat{n}s)B(\vec{x}-\hat{n}s) + b(\vec{x}-\hat{n}s)B_{\rm source}(\vec{x}-\hat{n}s,\hat{n})\} \exp\left\{-\int_0^s ds' \; c(\vec{x}-\hat{n}(s-s'))\right\} \\
&+ \sum_{n_w} \int d\Omega' \int_0^\infty d\ell \left| \frac{ds_{\rm WKB}[\ell,\vec{x}',\hat{\beta}_0^{n_w}]}{d\ell} \right| \\
&\times \left\{a(\vec{x}-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta}_0^{n_w}))B(\vec{x}-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta}_0^{n_w})) + b(\vec{x}-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta}_0^{n_w}))B_{\rm source}(\vec{x}-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta}_0^{n_w}\right. \\
&\times \exp\left\{-\int_0^\ell d\ell' \frac{c(\vec{x}-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta}_0^{n_w}) + \Delta\vec{x}'(\ell',\vec{x}',\hat{\beta}_0^{n_w}))}{b(\vec{x}-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta}_0^{n_w}) + \Delta\vec{x}'(\ell',\vec{x}',\hat{\beta}_0^{n_w}))}\right\} \\
&\times \left(e^\ell - 1\right) P_{\rm MS}\left(\Theta + 2n_w\pi, \alpha_\ell\right) .
\end{aligned}$$

At first glance this equation may appear complicated, but in fact it has a fairly straightforward structure. The term in braces after the angular integration represents the source term. The next term is the extinction coefficient integrated along the least attenuation path. This is then multiplied by $(\exp(\ell)-1)$ that represents the exponential growth of the number of rays with scattering length that are fluctuating around the least attenuation path. An exact raytrace algorithm would need an exponentially large number of rays to represent this and the virtue of the WKB approximation is that this effect is calculated rather than directly simulated. Finally, the $P_{\rm MS}$ term is the probability for a given ray to scatter through an angle Θ after traveling a scattering length ℓ.

To get some insight into the particular factors multiplying $P_{\rm MS}$ consider a uniform system with $B_{\rm source} = 0$, in which case

$$\begin{aligned}
\delta L(\vec{x},\hat{n}) &= B\left(a\int_0^\infty dse^{-cs} + \frac{a}{b}\int_0^\infty d\ell e^{-(c/b)\ell} \times (e^\ell - 1)\right) \\
&= B(\frac{a}{c} + 1 - \frac{a}{c}) \\
&= B
\end{aligned} \tag{42}$$

Here we see that the detected radiance is the black-body radiance and that a fraction a/c of the detected radiation was unscattered while a fraction b/c of the radiation was scattered. The factors that multiply $P_{\rm Ms}$ are the correct ones to give the proper ratio of scattered to unscattered radiation.

Equation 41 represents our final expression for the WKB approximation to multiple-scattering and is the central result of this document.

5 Sources of Light

This section describes the specific use of the WKB algorithm for collimated sources of light (solar light) and diffuse sources of light (earthshine and air radiance).

5.1 Collimated Sources: Solar Light

Two types of external sources are of primary interest: Sunlight and Earthshine. This section gives explicit expression for the source term $B_{source}(\vec{x},\hat{n})$ for sunshine, and the next section does the same for earthshine.

The sun is typically modeled as a collimated source, i.e.

$$L^{\infty}_{source}(\vec{x},\hat{n}) = H_{sun}\ \delta(\hat{n}-\hat{n}_{sun})\ \exp\{-\tau_{atmos}(\vec{x},\hat{n}_{sun})\}\ , \tag{43}$$

where H_{sun} is the irradiance of the sun at the top of the atmosphere, $\hat{n}_{sun}$ is the direction of the sun, and $\tau_{atmos}(\vec{x},\hat{n})$ is the atmospheric attenuation of the sunlight from the top of the atmosphere to the cloud, and is essentially constant over the spatial scales of the clouds we will model. The source term is then

$$B_{source}(\vec{x},\hat{n}) = H_{sun}\ \exp\{-\tau_{atmos}(\vec{x},\hat{n}_{sun})\}\ P(\hat{n}\cdot\hat{n}_{sun})\ \exp\left\{-\int_0^{\infty} ds\ c(\vec{x}-\hat{n}_{sun}s)\right\} \tag{44}$$

This quantity can be calculated with relative efficiency for particular clouds and sun directions. Starting at the cloud cells on faces directly exposed to the sun, a ray can burrow through each cell in the direction $\hat{n}_{sun}$, accumulating the quantity

$$\exp\left\{-\int_0^{\infty} ds\ c(\vec{x}-\hat{n}_{sun}s)\right\} \tag{45}$$

at each cloud cell along the ray. An 3D array can store these values for use during the main raytrace calculation.

5.2 Diffuse Sources: Earthshine

Diffuse sources are more difficult to treat because they do not allow the simple mathematical manipulation of section 5.1. If we were to try and evaluate B_{source} numerically for the earthshine, the computational load would be overwhelming because of the need to evaluate the solid angle integration of the attenuated earthshine at each point in the cloud volume.

Instead, we treat diffuse sources as a boundary value problem on the exterior of the cloud. Assume that $L_{\rm Earth}$ is known at each point on the cloud surface and we want to calculate $\delta L(\vec{x},\hat{n})$. We can use the evolution operator to directly evolve the radiance at the surface to the point $\vec{x}$. For each value of $\hat{n}'$ and n_w there is a value $\ell_{\rm boundary}(\hat{n}',n_w)$ for which the path $\vec{x}-\Delta\vec{x}'(\ell_{\rm boundary};\ell_{\rm boundary})$ is on the boundary of the cloud. Integrating over all angles then gives for the contribution to δL from the air radiance

$$\begin{aligned}\delta L(\vec{x},\hat{n}) &= \exp\left\{-\int_0^{s_{\text{boundary}}} ds'\ c(\vec{x}-\hat{n}(s-s'))\right\} L_{\text{Earth}}(\vec{x}-s_{\text{boundary}}\hat{n},\hat{n}) \\ &+ \sum_{n_w}\int d\Omega'\ \exp\left\{-\int_0^{\ell_{\text{boundary}}} d\ell' \frac{c(\vec{x}-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta}_0^{n_w})+\Delta\vec{x}'(\ell',\vec{x}',\hat{\beta}_0^{n_w}))}{b(\vec{x}-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta}_0^{n_w})+\Delta\vec{x}'(\ell',\vec{x}',\hat{\beta}_0^{n_w}))}\right\} \\ &\times \left(e^{\ell_{\text{boundary}}}-1\right) P_{\text{MS}}\left(\Theta+2n_w\pi,\alpha_{\ell_{\text{boundary}}}\right) L_{\text{Earth}}(\vec{x}-\Delta\vec{x}'(\ell,\vec{x}',\hat{\beta}_0^{n_w}),\hat{n}')\ . \qquad (46)\end{aligned}$$

6 Summary

Sections 4 and 5 have presented a complete raytrace algorithm based on the WKB approximation for propagating thermal, solar, and diffuse sky radiation through a cloud. In this algorithm a representative sample of rays is traced that correspond to the dominant paths taken by rays as they scatter through the cloud. These paths are nearly straight when b is small and highly curved when b is large. All other paths are dealt with by an analytical integration with respect to these paths. Multiple scattering reflects itself in this analytical formula through an ℓ dependent phase function that representing the fact that scattering becomes increasingly isotropic after a large number of scatterings and an ℓ dependent factor that represents the fact that the number of rays increases as the number of scatterings increase.

A subsequent document will describe the actual numerical implementation of these algorithms and present results.

References

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