Angular Smoothing and Spatial Diffusion from the Feynman Path Integral Representation of Radiative Transfer

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Abstract

The propagation kernel for time dependent radiative transfer is represented by a Feynman Path Integral (FPI). The FPI is approximately evaluated in the spatial-Fourier domain. Spatial diffusion is exhibited in the kernel when the approximations lead to a gaussian dependence on the Fourier domain wave vector. The approximations provide an explicit expression for the diffusion matrix. They also provide an asymptotic criterion for the self-consistency of the diffusion approximation. The criterion is weakly violated in the limit of large numbers of scattering lengths. Additional expansion of higher-order terms may resolve whether this weak violation is significant.

Key words:
Feynman path integral, diffusion, stationary phase, steepest descents, propagation kernel, computer graphics

1. Introduction

Simulations and imagery of smoke, water, clouds, and other natural phenomena are routinely generated in the computer graphics industry. The imagery is generated primarily from a single scattering approximation, although it can deviate from that for artistic purposes. Multiple scattering is very desirable, but the existing algorithms in use are very ad hoc, not visually good, not sufficiently flexible, and inefficient in the workflow. The algorithm in this paper is the basis for a new software tool addressing all of those limitations (although still with artistic deviations added to it). It has the benefit of beginning with a strong scientific footing in the radiative transfer equation, then is approximated only as much as necessary for the computer graphics application. Also, the algorithm is very flexible for added artistic deviations, and works very well in visual effects production. In this approach, the algorithm should ideally achieve both quasi-ballistic and diffusive regimes of transport behavior, and transition gracefully between the two.

The relationship between radiative transfer and its approximation by diffusion has been of intense interest and application for quite some time. In an extensive review by Davis and Marshak [1], the known methods to relate diffusion and radiative transfer are sorted into two categories. One is a constitutive approach, using Fick's Law to relate the scalar flux to vector flux; the other, including asymptotics, is an approximation of radiative transfer in the limit of many scattering events. Both approaches address the extreme of completely diffuse transport, but do not provide a description of how the multiple scattering transitions into diffusion. It would also be very useful to have a framework or model that characterizes the transition from the extreme of near-ballistic behavior to the diffusive regime in scattering media.

In this paper the relationship is analyzed in a new way. The radiative transfer problem is formulated in terms of a Feynman Path Integral (FPI), which then serves as the starting point for approximations leading to both angular smoothing and spatial diffusion. Radiative transfer has previously been evaluated in terms of FPIs and variations of it [2, 3, 4, 5], for applications in ocean optics [6], medical imaging [7], and computer graphics [8, 9]. Each application approximated the FPI in accordance with the application's needs, for example using the the small angle approximation for ocean and tissue optics. But none of them systematically attempt approximations which could potentially be valid

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in a broad range of applications including diffusive and non-diffusive settings. The approximations in the present paper are less restrictive than those of previous works, and apply across a broader range of problems. In particular, the approximation process used here produces an asymptotic relation for the self-consistency of the procedure.

The approximation is in three steps. The first is an approximation of the functional integral over "momentum," which is defined here as the Fourier conjugate of the increment in direction due to scattering. The result is achieved via a standard stationary phase procedure. The second is approximation of the functional integral over paths through the scattering medium via a steepest descents procedure. These first two steps produce explicit results which are not limited to a diffusive regime. The third and final approximation seeks a simplification that exhibits spatial diffusion. This simplification is the source of the asymptotic relation criterion for diffusion.

The remainder of this paper is organized as follows. In Section 2 the FPI for radiative transfer is examined in preparation for applying approximations. Section 3 discusses the energy conservation property of the radiative transfer propagation kernel, and how that property is preserved in the approximation process. The stationary phase and steepest descents approximations are carried out in Section 4. Spatial diffusion is exhibited in Section 5 after the asymptotic relation is defined and applied to the approximated kernel. The diffusion result is examined for self-consistency with the asymptotic relation as well. Speculation about further research into the mechanism for diffusion is contained in Section 6. Conclusions are presented and discussed in Section 7.

For completeness, the particular version of the FPI used in this paper is derived in Appendix A using the same character of argument used in standard derivations of the FPI in quantum mechanics [10].

Finally, it should be noted that all of the calculations and results for this approximation of the FPI for radiative transfer are shown for the special case of a medium that is uniform with infinite extent. By slightly rephrasing the FPI, a medium with arbitrary spatial variability can be handled. One of the biggest changes in the outcome is that the number of scattering lengths plays a much more important role in sorting out various paths that contribute to the stationary phase and steepest descents approximations. This topic will be the subject of future publications.

2. Feynman Path Integral Representation

The Feynman Path Integral (FPI) representation applies to the propagation kernel for time dependent radiative transfer. The radiance distribution L satisfies the equations

$$\left\{\frac{\partial}{\partial s} + \hat{\mathbf{n}}\cdot\nabla + c\right\} L(s,\mathbf{x},\hat{\mathbf{n}}) \;=\; b\int d\Omega(\hat{\mathbf{n}}')\, P(\hat{\mathbf{n}},\hat{\mathbf{n}}')L(s,\mathbf{x},\hat{\mathbf{n}}') + S(s,\mathbf{x},\hat{\mathbf{n}}), \tag{1}$$

$$L(0,\mathbf{x},\hat{\mathbf{n}}) \;=\; 0, \tag{2}$$

where c and b are the total and scattering extinction coefficients respectively, $a = c - b$ is the absorption coefficient, P is the normalized phase function, and S is a light source. The time s is in units of length after scaling by the light velocity. In terms of a propagation kernel, the explicit integral form of (1) is

$$L(s,\mathbf{x},\hat{\mathbf{n}}) = \int_0^s ds' \int d^3x' \int d\Omega(\hat{\mathbf{n}}')\, G(s-s',\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}')S(s',\mathbf{x}',\hat{\mathbf{n}}') \tag{3}$$

where the kernel G satisfies the initial value problem

$$\left\{\frac{\partial}{\partial s} + \hat{\mathbf{n}}\cdot\nabla + c\right\} G(s,\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}') \;=\; b\int d\Omega(\hat{\mathbf{n}}'')\, P(\hat{\mathbf{n}},\hat{\mathbf{n}}'')G(s,\mathbf{x},\hat{\mathbf{n}}'';\mathbf{x}',\hat{\mathbf{n}}') \tag{4}$$

$$G(0,\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}) = \delta(\mathbf{x}-\mathbf{x}')\delta(\hat{\mathbf{n}}-\hat{\mathbf{n}}'). \tag{5}$$

This integro-differential formulation of the radiative transfer problem also applies to steady-state problems, which can be viewed as a long-time limit of the time-dependent problem. A common application is that the source is always on and time-independent. Taking the long time limit, $s \to \infty$, the time-independent radiance can be expressed with this propagation kernel as

$$L(\mathbf{x},\hat{\mathbf{n}}) = \int d^3x' \int d\Omega(\hat{\mathbf{n}}')\, K(\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}')S(\mathbf{x}',\hat{\mathbf{n}}') \tag{6}$$

where

$$K(\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}') = \int_0^\infty ds\, G(s,\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}'). \tag{7}$$

The propagation kernel G has an exact formal expression in terms of a Feynman Path Integral. This expression is derived in the appendix. The path integral explicitly examines every possible path of length s from the "starting point" $\mathbf{x}'$ and "starting direction" $\hat{\mathbf{n}}'$, which ends at the point $\mathbf{x}$ and direction $\hat{\mathbf{n}}$. Each path is characterized by its unit tangent vector $\hat{\beta}(s')$ at each point $0 \le s' \le s$ along the path. Because each path starts at $\mathbf{x}'$ and ends at $\mathbf{x}$, the FPI must only include paths which satisfy

$$\mathbf{x} - \mathbf{x}' = \int_0^s ds'\, \hat{\beta}(s'). \tag{8}$$

Each path contributes a weight factor W in the FPI. The weight is related to how much scattering occurs along the path. The amount of scatter depends on the curvature of the path, $\kappa(s')$, where

$$d\hat{\beta}(s')/ds' = \kappa(s')\hat{\mathbf{N}}(s'), \tag{9}$$

and $\hat{\mathbf{N}}$ is the normal to the path in the sense of Frenet-Serret curves. Using the curvature in this way is a choice of convenience to simplify notation. It is not meant to imply that the paths have smooth tangents, and the FPI obtained below does not require them. The scattering weight also depends on the phase function and scattering coefficient. The FPI derivation in the appendix introduces a "momentum" variable $\mathbf{p}(s')$ at each point of the path, which is integrated over all possible momentum configurations. The weight is explicitly expressed as the functional integral[1]

$$\begin{aligned} W(s,\kappa) = & \int [d\mathbf{p}] \exp\left\{ i \int_0^s ds' \mathbf{p}(s') \cdot \hat{\mathbf{N}}(s')\kappa(s') \right\} \\ & \times \exp\left\{ \int_0^s ds'\, b\left(\tilde{Z}(|\mathbf{p}(s')|) - 1 \right) \right\} \end{aligned} \tag{10}$$

in which the phase function has been converted to its pseudo-Fourier transform via the relationship

$$P(\hat{\mathbf{n}},\hat{\mathbf{n}}') = \int \frac{d^3p}{(2\pi)^3}\, \tilde{Z}(|\mathbf{p}|)\, \exp\left(i\mathbf{p}\cdot(\hat{\mathbf{n}} - \hat{\mathbf{n}}')\right). \tag{11}$$

Using the weights W and the restrictions on the path, the full FPI for the kernel is

$$\begin{aligned} G(s,\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}') = \; & e^{-as} \int [d\hat{\beta}]\, W(s,\kappa)\, \delta\left(\mathbf{x} - \mathbf{x}' - \int_0^s ds'\, \hat{\beta}(s') \right) \\ & \times\; \delta(\hat{\beta}(0) - \hat{\mathbf{n}}')\delta(\hat{\beta}(s) - \hat{\mathbf{n}}) \end{aligned} \tag{12}$$

where $a = c - b$ is the absorption coefficient.

For the integral over paths, the hard constraint in (8) is difficult to enforce in an approximation scheme. Instead, we can also express the kernel in terms of a spatial Fourier transform

$$G(s,\mathbf{x},\hat{\mathbf{n}};\mathbf{x}',\hat{\mathbf{n}}') = \int \frac{d^3k}{(2\pi)^3}\, \tilde{G}(s,\mathbf{k},\hat{\mathbf{n}},\hat{\mathbf{n}}')\, \exp\left(i\mathbf{k}\cdot(\mathbf{x} - \mathbf{x}')\right) \tag{13}$$

and the Fourier amplitude has the FPI

$$\begin{aligned} \tilde{G}(s,\mathbf{k},\hat{\mathbf{n}},\hat{\mathbf{n}}') = \; & e^{-as} \int [d\hat{\beta}]\, W(s,\kappa)\, \exp\left(-i \int_0^s ds' \mathbf{k}\cdot\hat{\beta}(s') \right) \\ & \times\; \delta(\hat{\beta}(0) - \hat{\mathbf{n}}')\delta(\hat{\beta}(s) - \hat{\mathbf{n}}). \end{aligned} \tag{14}$$

In this form the enforcement of the constraint in (8) is relegated to the Fourier integral over the spatial wavevector $\mathbf{k}$. However, approximations carried out in Section 4.3 relax enforcement of (8).

[1]Throughout this paper, $[d\cdot]$ denotes a differential element defined with respect to a natural measure; see Appendix A.

3. Normalization via Energy Conservation

In evaluating approximate expressions for $\tilde{G}$, energy conservation serves as a useful normalization criterion. In particular, there is the specific property

$$\int d^3x \int d\Omega(\hat{\mathbf{n}})\, G(s, \mathbf{x}, \hat{\mathbf{n}}; \mathbf{x}', \hat{\mathbf{n}}') \equiv \int d\Omega(\hat{\mathbf{n}})\, \tilde{G}(s, 0, \hat{\mathbf{n}}, \hat{\mathbf{n}}') \;=\; \exp(-as). \tag{15}$$

This conservation property will be enforced in the approximations below as a form of normalization of the final expressions emerging from the approximation process. This normalization replaces evaluation of the normalization of the functional integrals.

4. Approximate Evaluation

The attack for generating an approximate expression for $\tilde{G}$ has two major steps:

1. An approximation (via stationary phase) for the weight W for any path. This is accomplished in Section 4.1.
2. An approximation (via steepest descents) of the functional integral over paths $\hat{\beta}$. This is accomplished in Section 4.2.

To make this process concrete, the phase function is fixed as a unit-normalized gaussian shape:

$$P(\hat{\mathbf{n}}, \hat{\mathbf{n}}') = \frac{2N_p}{(2\pi\mu)^{3/2}} \exp\left(\frac{\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' - 1}{\mu}\right) \tag{16}$$

where the cosine of the scattering angle Θ, $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'$, varies between $+1$ and -1, and N_p is the normalization constant

$$N_p = \frac{\sqrt{\pi\mu/2}}{1 - \exp(-2/\mu)}. \tag{17}$$

The parameter $\mu > 0$ can be viewed as the mean square width of the peak of the phase function since, in the limit of $\Theta \ll 1$, we have $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' - 1 \approx -\Theta^2/2$. This also makes clear the gaussian nature of the phase function. Apart from this small-μ limit, which is familiar to the small-angle approximation, the large-μ limit leads to

$$P(\hat{\mathbf{n}}, \hat{\mathbf{n}}') \approx \frac{1}{4\pi}\left(1 + \frac{\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'}{\mu}\right), \tag{18}$$

which is the familiar form in photon diffusion theory. Generally speaking, the asymmetry factor of the adopted phase function is

$$\langle \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' \rangle = \coth(1/\mu) - \mu. \tag{19}$$

For $\mu \gg 1$, this yields $\langle \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' \rangle \approx 1/3\mu$ hence only a small deviation from isotropic scattering, which will promote the diffusive transport.

With the choice of phase function in (16)–(17), we have

$$\tilde{Z}(|\mathbf{p}|) = N_p \exp\left(-\mu|\mathbf{p}|^2/2\right). \tag{20}$$

Much of the approximation process carried out in the sections below can be done with more complex phase functions, for example, phase functions composed of a mix of gaussian phase functions.

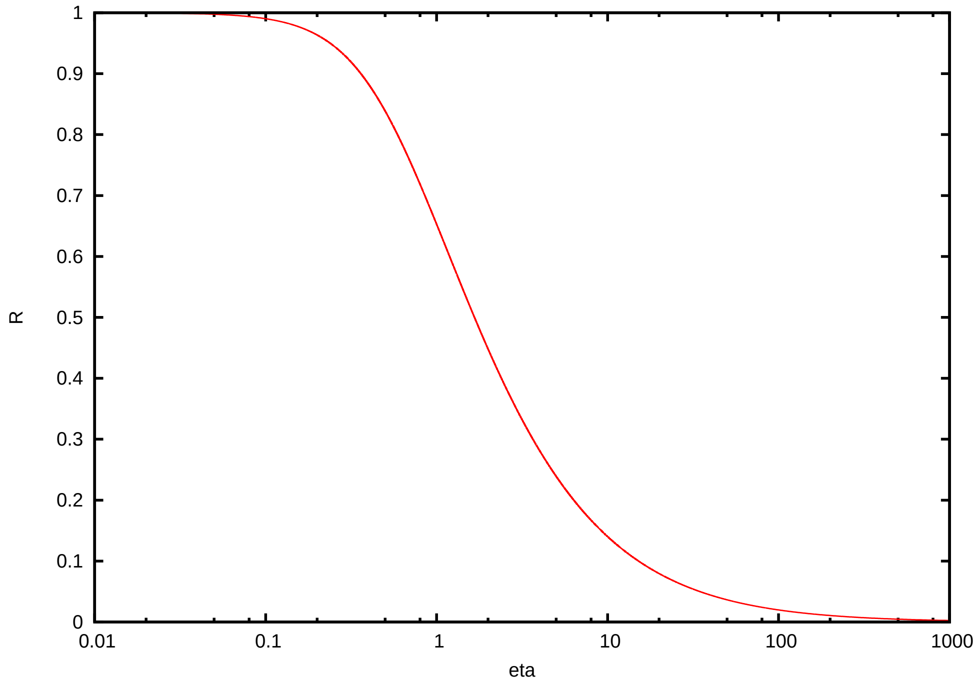

Figure 1: R as a function of η.

4.1. Stationary Phase Evaluation of the Weight W

The stationary phase approximation is motivated by writing the functional integral for W using a complex "action" A:

$$W(s, \kappa) = \int [d\mathbf{p}] \exp\{-A(\mathbf{p})\} \tag{21}$$

with

$$A(\mathbf{p}) = \int_0^s ds' \left\{-i\mathbf{p}(s') \cdot \hat{\mathbf{N}}(s')\kappa(s') \; - \; b\left(\tilde{Z}(|\mathbf{p}(s')|) - 1\right)\right\}. \tag{22}$$

Extremizing this functional requires that $\mathbf{p}$ be rotated to the imaginary axis [11]. Using the gaussian phase function above, the stationary point is

$$\mathbf{p}_0 = i\hat{\mathbf{N}}\frac{\kappa}{b\mu N_p}R \tag{23}$$

where R satisfies the implicit equation

$$R = \exp(-\eta^2 R^2) \tag{24}$$

where

$$\eta^2 = \frac{\kappa^2}{2\mu b^2 N_p^2}. \tag{25}$$

The quantity $R(\eta)$ varies in the range $(0, 1]$ and is monotonic in η, which itself depends only on the scattering coefficient b and the phase function parameter μ; it is plotted in Fig. 1. With this stationary point, the exponential is

$$A(\mathbf{p}_0) = \int_0^s ds' \; b\left\{1 + N_p\left(2\eta^2(s')R(s') - 1/R(s')\right)\right\} \tag{26}$$

This stationary point minimizes A, and the stationary value is real.

Note that this minimization is analogous to the semiclassical approximation in quantum mechanics, where the parameter $\hbar$ is assumed to be small to drive the approximation process. The analogous parameter in this situation is $1/b$, and the approximation is driven by assuming b is large in some sense.

For a full leading order stationary phase approximation, the next step would be to calculate the quadratic expansion around the stationary point and evaluate the functional integral for that expansion. A more useful approach in this case is to keep only the stationary value, and leave the remaining factor to be fixed by the energy conservation normalization discussed in Section 3. Consequently, the result of our stationary phase evaluation is

$$W(s,\kappa) = W_0 \exp(-A(\mathbf{p}_0)) \tag{27}$$

and the value of W_0 will be fixed (in Section 4.3) by enforcing energy conservation.

4.2. Steepest Descents Evaluation of the Kernel

After the stationary phase evaluation of $W(s,\kappa)$, the FPI for the kernel is

$$\begin{aligned}\tilde{G}(s,\mathbf{k},\hat{\mathbf{n}},\hat{\mathbf{n}}') = \quad & e^{-as} W_0 \quad \int [d\hat{\beta}] \exp\left(-A(\mathbf{p}_0) \;- i\int_0^s ds'\mathbf{k}\cdot\hat{\beta}(s')\right) \\ & \times \;\; \delta(\hat{\beta}(0)-\hat{\mathbf{n}}')\delta(\hat{\beta}(s)-\hat{\mathbf{n}}).\end{aligned} \tag{28}$$

To find the extremum path, $\hat{\beta}_0(s')$, of the functional integral over paths $\hat{\beta}$, we can look at the variational derivative of A with respect to $\hat{\beta}$. All of the dependence on $\hat{\beta}$ comes through the dependence on $\kappa^2(s')$ in η and R, so the variational derivative is

$$\frac{\delta A}{\delta\hat{\beta}(s')} = \frac{\delta A}{\delta\kappa^2(s')}\frac{d\kappa^2(s')}{d\hat{\beta}(s')}. \tag{29}$$

In this situation, minimizing A is the same as minimizing κ^2 with respect to $\hat{\beta}$. The steepest descents procedure selects the smoothest, least curved path that satisfies the boundary conditions. For an explicit solution of this minimization, the path tangents can be decomposed into a spherical representation using the orthogonal coordinate system $\{\hat{\mathbf{n}}',\hat{\mathbf{t}},\hat{\mathbf{r}}\}$, with

$$\begin{aligned}\hat{\mathbf{t}} &= \frac{\hat{\mathbf{n}}'\times\hat{\mathbf{n}}}{|\hat{\mathbf{n}}'\times\hat{\mathbf{n}}|} \\ \hat{\mathbf{r}} &= \frac{\hat{\mathbf{n}}'\times\hat{\mathbf{t}}}{|\hat{\mathbf{n}}'\times\hat{\mathbf{t}}|}\end{aligned}$$

from which $\hat{\beta}(s')$ has the form

$$\hat{\beta}(s') = \cos(g(s'))\left[\hat{\mathbf{n}}'\cos(f(s')) + \hat{\mathbf{r}}\sin(f(s'))\right] + \sin(g(s'))\hat{\mathbf{t}}. \tag{30}$$

To satisfy the boundary conditions, the otherwise-arbitrary functions f and g must have the boundary conditions

$$\begin{aligned}g(0) &= 2\pi n \\ g(s) &= 2\pi m \\ f(0) &= 2\pi l \\ f(s) &= \Theta + 2\pi k\end{aligned}$$

where k,l,m,n are integers, and we recall that $\Theta = \cos^{-1}(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')$. The functions which minimize curvature are $g_0(s') = 0$ and $f_0(s') = \Theta s'/s$. In addition, this path has constant curvature $\kappa_0 = \Theta/s$, which means that η, and therefore R, are also constant for this path, and

$$A_0 = bs\left\{1 + N_p\left(2\eta_0^2(s')R_0(s') - 1/R_0(s')\right)\right\} \tag{31}$$

with

$$\eta_0^2 = \frac{\Theta^2}{2\mu(bs)^2 N_p^2}, \tag{32}$$

$$R_0 = \exp(-\eta_0^2 R_0^2). \tag{33}$$

Having established the minimizing path $\hat{\beta}_0$, fluctuations around that path come from setting $f = f_0 + \delta f$ and $g = \delta g$, and expanding the exponential integrand up to quadratic in δf and δg. The expansion of A has only quadratic terms, but linear and quadratic terms come from $\int \mathbf{k} \cdot \hat{\beta}$. However, there are no terms proportional to $\delta f\ \delta g$, so the functional integrals over δf and δg are separable. Each of those separated functional integrals has the form

$$\begin{aligned} I(\mathbf{J}, M) &= \int [d\rho]\ \delta(\rho(0))\ \delta(\rho(s)) \\ &\times\ \exp\left\{-i \int_0^s ds'\ (\mathbf{k} \cdot \mathbf{J}(s'))\ \rho(s')\right\} \\ &\times\ \exp\left\{-\frac{1}{2} \int_0^s ds'\ \rho(s')\ \Delta_{M\mathbf{k}}^{-1}\ \rho(s')\right\}, \end{aligned} \tag{34}$$

where M is a different parameter for each of the two integrals, as shown below, and the differential operator $\Delta_{M\mathbf{k}}^{-1}$ is

$$\Delta_{M\mathbf{k}}^{-1} = \left(-\frac{R_0}{\mu b N_p} \frac{d^2}{ds'^2} - \frac{R_0}{\mu b N_p} M^2 - i\mathbf{k} \cdot \hat{\beta}_0(s')\right) \tag{35}$$

the kernel is

$$\begin{aligned} \tilde{G}(s, \mathbf{k}, \hat{\mathbf{n}}, \hat{\mathbf{n}}') &\approx e^{-as}\ W(s, \kappa_0)\ \exp\left(-i \int_0^s ds'\ \mathbf{k} \cdot \hat{\beta}_0(s')\right) \\ &\times\ I(\hat{\beta}_0, 0)\ I(\hat{\mathbf{t}}, \kappa_0/\sqrt{2}) \end{aligned} \tag{36}$$

Formally, the functional integral $I(\mathbf{J}, M)$ has the evaluation

$$\begin{aligned} I(\mathbf{J}, M) &= \exp\left\{-\frac{1}{2} \int_0^s ds' \int_0^s ds''\ (\mathbf{k} \cdot \mathbf{J}(s'))\ \Delta_{M\mathbf{k}}(s', s'')\ (\mathbf{k} \cdot \mathbf{J}(s''))\right\} \\ &\times\ (\mathrm{Det}\Delta_{M\mathbf{k}})^{1/2} \end{aligned} \tag{37}$$

This expression for the kernel has a complex dependence on the spatial wavevector $\mathbf{k}$ because of the Green's function $\Delta_{M\mathbf{k}}$. For details the reader is referred to Appendix B.

4.3. Normalization

Using the energy conservation property of the kernel on the expression in (36), a normalization condition for W_0 is generated. Setting $\mathbf{k} = 0$ and integrating over the unit sphere, the normalization is

$$W_0 = \left(\int d\Omega(\hat{\mathbf{n}})\ (\mathrm{Det}\Delta_{00})^{1/2}\ \exp(-A_0)\ (\mathrm{Det}\Delta_{(\kappa_0/\sqrt{2})0})^{1/2} \right)^{-1}. \tag{38}$$

Note that the Green's functions Δ_{M0} have relatively simple expressions:

$$\Delta_{M0}(s', s'') = \frac{\mu b N_p}{R_0} \frac{\sin(Ms^<) \sin(M(s - s^>))}{M \sin(Ms)}, \tag{39}$$

where $s^> = \max(s', s'')$ and $s^< = \min(s', s'')$. So do their determinants:

$$\mathrm{Det}\Delta_{M0} = \frac{R_0 M}{\mu b N_p \sin(Ms)}. \tag{40}$$

5. Spatial Diffusion

The approximation for $\tilde{G}$ arrived at in (36) is more complex than a simple spatial diffusion. Spatial diffusion would be recognized if the result were gaussian in the wavevector $\mathbf{k}$, with the gaussian width being the diffusion parameter. Indeed the quantities I in (37) are superficially gaussian in character, except that both the gaussian coefficient and the determinant depend on $\mathbf{k}$ also.

However we can ask whether there is an asymptotic regime under which this approximation for $\tilde{G}$ does become gaussian, and hence spatially diffusive. In this section we develop the criterion for such a regime, and evaluate its self-consistency. First, the asymptotic regime is established, and after analyzing the spatial diffusion produced in that regime we establish the nature of its consistency.

To arrive at a gaussian dependence in $\tilde{G}$ on $\mathbf{k}$, we must perform perturbation expansions in $\mathbf{k}$ on $\Delta_{M\mathbf{k}}$. This produces the criterion for the asymptotic regime. There must be an asymptotic relation between the differential term of $\Delta_{M\mathbf{k}}$ and the one proportional to $\mathbf{k}$. The term in (35) proportional to $\mathbf{k}$ is asymptotically small if

$$|\mathbf{k}| \ll \frac{R_0}{\mu b s^2 N_p}. \tag{41}$$

Rearranging terms, this relation is also

$$\frac{|\mathbf{k}|}{b} \ll \frac{R_0}{\mu (bs)^2 N_p}. \tag{42}$$

This asymptotic condition says that this perturbation process is consistent if the smallest length scale, ℓ, of the light field satisfies

$$\ell\, b \gg \frac{\mu N_p (bs)^2}{R_0}. \tag{43}$$

Using this statement of the criteria for a spatially diffusive asymptotic region, we construct explicitly the diffusive form of $\tilde{G}$. This provides an explicit value for the diffusion matrix, which in turn provides the smallest scale of the diffusive light field. Finally, the scale feeds back into the asymptotic regime criteria to evaluate its self-consistency.

The quantities $I(\mathbf{J}, M)$ are already gaussian with just the lowest term of the expansion, Δ_{M0}, applied in the exponential. So these quantities become

$$\begin{aligned} I(\mathbf{J}, M) \quad &\approx \quad \exp\left\{-\frac{1}{2}\int_0^s ds' \int_0^s ds''\, (\mathbf{k}\cdot\mathbf{J}(s'))\, \Delta_{M0}(s', s'')\, (\mathbf{k}\cdot\mathbf{J}(s''))\right\} \\ &\times \quad (\mathrm{Det}\Delta_{M\mathbf{k}})^{1/2}. \end{aligned} \tag{44}$$

However, a longer expansion is necessary for the determinant factor. Using identities relating determinant and traces of operators, the determinant can be exactly rearranged as

$$\begin{aligned} (\mathrm{Det}\Delta_{M\mathbf{k}})^{1/2} \quad &= \quad (\mathrm{Det}\Delta_{M0})^{1/2}\, (\mathrm{Det}(1 - i\Delta_{M0}\mathbf{k}\cdot\hat{\beta}_0)^{-1/2} \\ &= \quad (\mathrm{Det}\Delta_{M0})^{1/2}\, \exp\left(-\frac{1}{2}\mathrm{Tr}\ln(1 - i\Delta_{M0}\mathbf{k}\cdot\hat{\beta}_0)\right). \end{aligned} \tag{45}$$

By applying the asymptotic criteria, the trace term can be Taylor expanded in the natural logarithm up to quadratic order in $\mathbf{k}$, giving

$$\begin{aligned} (\mathrm{Det}\Delta_{M\mathbf{k}})^{1/2} \quad &\approx \quad (\mathrm{Det}\Delta_{M0})^{1/2}\, \exp\left(-i\frac{1}{2}\mathrm{Tr}(\Delta_{M0}\mathbf{k}\cdot\hat{\beta}_0)\right) \\ &\times \quad \exp\left(-\frac{1}{2}\mathrm{Tr}\left(\Delta_{M0}\mathbf{k}\cdot\hat{\beta}_0\, \Delta_{M0}\mathbf{k}\cdot\hat{\beta}_0\right)\right). \end{aligned} \tag{46}$$

Now all factors and terms in the exponential factor in the expression for $\tilde{G}$ are up to 2nd-order (i.e., gaussian) in $\mathbf{k}$. Assembling them, $\tilde{G}$ has the form

$$\tilde{G}(s, \mathbf{k}, \hat{\mathbf{n}}, \hat{\mathbf{n}}') \approx e^{-as}\, G_0(s, \Theta)\, \exp\left(-\mathbf{k}\cdot\mathbf{X}(s) - \frac{1}{2}\mathbf{k}\cdot\mathcal{B}(s)\cdot\mathbf{k}\right) \tag{47}$$

where $\mathbf{X}$ and $\mathcal{B}$ are discussed below, and

$$G_0(s,\Theta) = \frac{\exp(-A_0(s))\ (\mathrm{Det}\Delta_{00})^{1/2}\ (\mathrm{Det}\Delta_{(\kappa_0/\sqrt{2})0})^{1/2}}{\int d\Omega(\hat{\mathbf{n}})\exp(-A_0(s))\ (\mathrm{Det}\Delta_{00})^{1/2}\ (\mathrm{Det}\Delta_{(\kappa_0/\sqrt{2})0})^{1/2}}. \tag{48}$$

5.1. Scatter Delay $\boldsymbol{X}$

Just by the boundary conditions on the paths through the volume, the steepest descents path $\hat{\beta}_0$ is curved, and the length of the path is s. The endpoint of this path is

$$\mathbf{x}' \ +\ \int_0^s ds'\,\hat{\beta}_0(s'). \tag{49}$$

This path has the same form as the "classical paths" in Perelman's photon migration framework [4]. However, the path integral approximation here contains additional deviations arising from scattering, so that the endpoint is

$$\mathbf{x}' \ +\ \mathbf{X}(s) \tag{50}$$

where

$$\mathbf{X}(s) \ =\ \int_0^s ds'\,\hat{\beta}_0(s')\left(1\ +\ \frac{1}{2}\left(\Delta_{00}(s',s') + \Delta_{(\kappa_0/\sqrt{2})0}(s',s')\right)\right). \tag{51}$$

We can evaluate these integrals with the help of (39). Letting

$$\xi = \frac{\mu N_p(\mu)}{R_0}\,bs, \tag{52}$$

we arrive at

$$\mathbf{X} = X_n\hat{\mathbf{n}} + X_r\hat{\mathbf{r}} \tag{53}$$

with

$$X_n(s) \ =\ s\frac{\sin\Theta}{\Theta}\left(1+\frac{\xi}{\Theta^2}\left(1-\frac{\Theta}{\sqrt{2}}\cot\left(\frac{\Theta}{\sqrt{2}}\right)\right)\right), \tag{54}$$

$$X_r(s) \ =\ s\frac{\cos\Theta-1}{\Theta}\left(1-\frac{\xi}{\Theta^2}\left(1-\frac{\Theta}{\sqrt{2}}\cot\left(\frac{\Theta}{\sqrt{2}}\right)\right)\right). \tag{55}$$

One way to think of this delay is to visualize what happens when laser pulses propagate through a turbid medium. The sharp pulse shape is altered and a tail develops of photons that have been delayed by scattering. This shifts the mean center of the profile backward from the original peak. The exiting profile of the pulse is asymmetric, with the front of the profile looking like the original pulse. In the asymptotic regime applied here, there is insufficient spatial detail to accurately model the original pulse shape (unless the original pulse was a gaussian, otherwise higher orders in $\mathbf{k}$ would be needed), and only the delay and broadening of the pulse are obtained.

5.2. Asymmetric Diffusion Matrix

The gaussian portion of $\tilde{G}$ is not characterized by a single broadening width, but instead by a broadening matrix, $\mathcal{B}$. This matrix is

$$\mathcal{B}(s) \ =\ \int_0^s dt\int_0^s dt'\ \left(\hat{\beta}_0(t)\hat{\beta}_0(t')\,\rho(t,t')\ +\ \hat{\mathbf{t}}\hat{\mathbf{t}}\,\Delta_{00}(t,t')\right) \tag{56}$$

and

$$\rho(t,t') = \Delta_{00}(t,t') + \frac{1}{2}(\Delta_{00}(t,t')\Delta_{00}(t',t) + \Delta_{(\kappa_0/\sqrt{2})0}(t,t')\Delta_{(\kappa_0/\sqrt{2})0}(t',t)) \tag{57}$$

This matrix is asymmetric with respect to the major axes. In particular, the direction perpendicular to the plane of the curved steepest descents path, $\hat{\mathbf{t}}$, has only a single scattering order in it and is due to just the leading order of the steepest descents approximation, while the plane of the path also has a quadratic term due to the scattering in paths near the steepest descents path. These nearby paths contribute to the determinant factors.

Using the form of the steepest descents path, this matrix evaluates to

$$\mathcal{B}(s) = \mathcal{B}_{nn}(s)\, \hat{\mathbf{n}}'\hat{\mathbf{n}}' \;+\; \mathcal{B}_{nr}(s)\, (\hat{\mathbf{n}}'\hat{\mathbf{r}} + \hat{\mathbf{r}}\hat{\mathbf{n}}') \;+\; \mathcal{B}_{rr}(s)\, \hat{\mathbf{r}}\hat{\mathbf{r}} \;+\; \mathcal{B}_{tt}(s)\hat{\mathbf{t}}\hat{\mathbf{t}} \tag{58}$$

with

$$\begin{aligned}
\mathcal{B}_{tt} &= \frac{\xi s^2}{12}, \\
\mathcal{B}_{ij} &= \frac{\xi s^2}{24\Theta^6}\, Q_{ij}(s) \quad (ij = nn,\ nr,\ rr), \\
Q_{nn} &= \xi\, b_{nn} + 12\Theta^4 - 6\Theta^3 \sin(2\Theta) - 96\Theta^2 \sin^4\left(\frac{\Theta}{2}\right), \\
Q_{nr} &= \xi\, b_{nr} - 12\Theta^3 \sin^2(\Theta) - 24\Theta^2 \sin(\Theta)(\cos(\Theta) - 1), \\
Q_{rr} &= \xi\, b_{rr} + 12\Theta^4 + 6\Theta^3 \sin(2\Theta) - 24\Theta^2 \sin^2\Theta, \\
b_{nn} &= 2\Theta^4 + 6\sqrt{2}\Theta^3 \cot\left(\frac{\Theta}{\sqrt{2}}\right) + \frac{3}{4}\Theta^2 \csc^2\left(\frac{\Theta}{\sqrt{2}}\right)\Big(\big(6 - 22\cos\big(\sqrt{2}\Theta\big)\big)\cos^2\Theta \\
&+ 3\cos(2\Theta) + 2\big(11\sin^2\Theta + 9\big)\cos\big(\sqrt{2}\Theta\big) - 28\sqrt{2}\sin\Theta \sin\big(\sqrt{2}\Theta\big)\cos\Theta - 5\Big) \\
&+ 48\sin^2\Theta - 30\Theta\cos\Theta\sin\Theta, \\
b_{nr} &= -3\Theta^2 \sin\Theta \csc^2\left(\frac{\Theta}{\sqrt{2}}\right)\Big(7\sqrt{2}\sin\Theta\sin\big(\sqrt{2}\Theta\big) + \cos\Theta\big(11\cos\big(\sqrt{2}\Theta\big) - 3\big) - 8\Big) \\
&- 30\Theta\sin^2\Theta - 48\sin\Theta(\cos\Theta - 1), \\
b_{rr} &= 2\Theta^4 + 6\sqrt{2}\Theta^3 \cot\left(\frac{\Theta}{\sqrt{2}}\right) + 192\sin^4\left(\frac{\Theta}{2}\right) + 30\Theta\sin\Theta\cos\Theta \\
&+ \frac{3}{4}\Theta^2\csc^2\left(\frac{\Theta}{\sqrt{2}}\right)\Big(11 + 3\cos(2\Theta) + \cos^2\Theta\big(6 - 22\cos(\sqrt{2}\Theta)\big) \\
&+ \cos(\sqrt{2}\Theta)\big(22\sin^2\Theta - 62\big) + \cos\Theta\big(64 - 28\sqrt{2}\sin\Theta\sin(\sqrt{2}\Theta)\big)\Big).
\end{aligned}$$

This matrix has three real eigenvalues. One of them is $\mathcal{B}_{tt}$, and the other two are the eigenvalues of the 2×2 symmetric matrix constructed from $\mathcal{B}_{nn}$, $\mathcal{B}_{nr}$ and $\mathcal{B}_{rr}$. These two eigenvalues, labelled $\mathcal{B}_+$ and $\mathcal{B}_-$, straddle $\mathcal{B}_{tt}$ in value, as can be seen in Fig. 2. The three eigenvalues are effectively diffusion coefficients for the spatial diffusion process. They also act as the measures of the spatial content of the light field.

5.3. Asymptotic Consistency

The consistency of the diffusion approximation that converts (36) to the gaussian form in (47) is based on the expectation that the spatial scales generated by the light field satisfy the asymptotic criterion in (43). Now that the diffusion scales have been calculated from the approximation, we can return to the criteria and verify or refute the consistency.

To proceed, the largest eigenvalue $\sqrt{\mathcal{B}_+}$ is chosen as the spatial scale of interest. Since these are asymptotic requirements, we take $\ell \sim \sqrt{\mathcal{B}_+}$ but not worry about exact numerical factors.

We examine consistency under two extremes of very small amount of scattering ($bs \to 0$) and large amounts of scattering ($bs \to \infty$). In the small amount of scattering regime, $\ell \sim s\sqrt{\xi}$, and the second asymptotic criterion is the balance

$$\left(\frac{\mu N_p}{R_0}\right)^{1/2} (bs)^{3/2} \gg \frac{\mu N_p}{R_0}\, (bs)^2, \tag{59}$$

which is a correct relationship as $bs \to 0$. In the opposite regime of large amounts of scattering, $\ell \sim s\xi$, the balance is

$$\frac{\mu N_p}{R_0}\, (bs)^2 \gg \frac{\mu N_p}{R_0}\, (bs)^2, \tag{60}$$

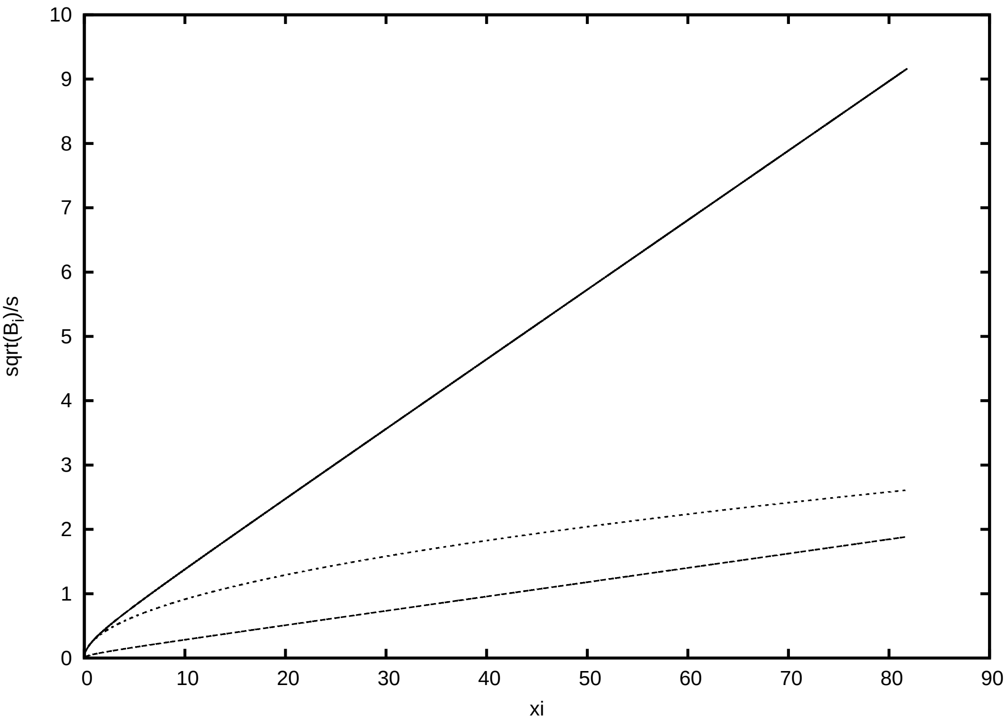

Figure 2: Diffusion matrix eigenvectors versus ξ, for the scattering angle $\Theta = \pi/2$. The eigenvectors, as displayed, are $\sqrt{\mathcal{B}_+}/s$ (solid), $\sqrt{\mathcal{B}_{tt}}/s$ (dot), and $\sqrt{\mathcal{B}_-}/s$ (dash).

which is not a correct asymptotic relationship for $bs \to \infty$. However, this is a "weak" violation in the sense that the balance is not reversed in direction. The two sides are similar in magnitude in the limit of large amounts of scattering. This is an indication that the expansion in small $|\mathbf{k}|$ may require more terms in order to form a leading order approximation. It also indicates that the criterion in (43) is probably insufficient to fully understanding the asymptotic expansion in the limit of large scattering lengths.

6. Does Spatial Diffusion Appear in the New FPI Approach?

The traditional definition of spatial diffusion for radiative transfer is that, in the limit of long times (hence many scattering events), irradiance F (angularly integrated L) satisfies

$$\frac{\partial F}{\partial s} \propto \nabla^2 F, \tag{61}$$

where the proportionality constant, $\mathcal{D}$, is the (coefficient of) diffusivity. An analysis of the present approximation shows that the behavior achieved here is

$$\frac{1}{s^2}\frac{\partial F}{\partial s} \propto \nabla^2 F. \tag{62}$$

To look at the long-time behavior of the irradiance, the Fourier integral of the spatial domain can be evaluated to produce the kernel:

$$G(s, \mathbf{x}, \hat{\mathbf{n}}; \mathbf{x}', \hat{\mathbf{n}}') \;\propto\; \det\left(\mathcal{B}(s)\right)^{-1/2}\; \exp\left(-\frac{1}{2}\left(\mathbf{x}-\mathbf{x}'-\mathbf{X}(s)\right)\cdot\mathcal{B}^{-1}(s)\cdot\left(\mathbf{x}-\mathbf{x}'-\mathbf{X}(s)\right)\right), \tag{63}$$

which has temporal and spatial derivatives:

$$\frac{\partial}{\partial s}G(s, \mathbf{x}, \hat{\mathbf{n}}; \mathbf{x}', \hat{\mathbf{n}}') \;=\; R(s)\, G(s, \mathbf{x}, \hat{\mathbf{n}}; \mathbf{x}', \hat{\mathbf{n}}'), \tag{64}$$

$$\nabla^2 G(s, \mathbf{x}, \hat{\mathbf{n}}; \mathbf{x}', \hat{\mathbf{n}}') \;=\; -S(s)\, G(s, \mathbf{x}, \hat{\mathbf{n}}; \mathbf{x}', \hat{\mathbf{n}}'), \tag{65}$$

with

$$S(s) \;=\; \mathrm{trace}(\mathcal{B}^{-1}(s)) - \left(\mathbf{x}-\mathbf{x}'-\mathbf{X}(s)\right)\cdot\mathcal{B}^{-2}(s)\cdot\left(\mathbf{x}-\mathbf{x}'-\mathbf{X}(s)\right), \tag{66}$$

$$R(s) \;=\; \frac{1}{2}\frac{\partial}{\partial s}\left(\mathrm{trace}(\mathcal{B}(s)) + \left(\mathbf{x}-\mathbf{x}'-\mathbf{X}(s)\right)\cdot\mathcal{B}^{-1}(s)\cdot\left(\mathbf{x}-\mathbf{x}'-\mathbf{X}(s)\right)\right). \tag{67}$$

The quantities $R(s)$ and $S(s)$ are plotted in Fig. 3 as a function of s, for multiple scattering angles Θ. These plots assume that $s \gg |\mathbf{x}-\mathbf{x}'|$, which allows us to ignore $\mathbf{x}-\mathbf{x}'$ as being small. In the long time limit, both quantities become independent of angle. Consequently, at long times the irradiance F satisfies

$$\frac{\partial}{\partial s}F(s) \;=\; R(s)\, F(s) \tag{68}$$

$$\nabla^2 F(s) \;=\; -S(s)\, F(s) \tag{69}$$

However, S/R is not a constant at long times, but more like $S(s) \sim R(s)/s^2$. Consequently, this approximation does not exhibit typical diffusion at long times.

As we see it, the current approximation is not a complete leading order solution, in the sense that there are multiple extremum paths. In (30), the unit vector was expressed in terms of two functions $g(s')$ and $f(s')$ which had to satisfy boundary conditions and be extremal paths. There is a set of extremal paths

$$f_m(s') = (2\pi m + \Theta)s'/s \tag{70}$$

and for the remainder of the paper we used the simplest one $m = 0$. But, in general, all integers m qualify as extremal paths. For these paths the integer m counts the number and orientation of loops in the path in the plane of $\hat{\mathbf{n}}'$-$\hat{\mathbf{r}}$. If we include these paths in the approximation, then the full expression for the kernel will look like

$$G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \propto \sum_{m=-\infty}^{\infty} G_m(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}'), \tag{71}$$

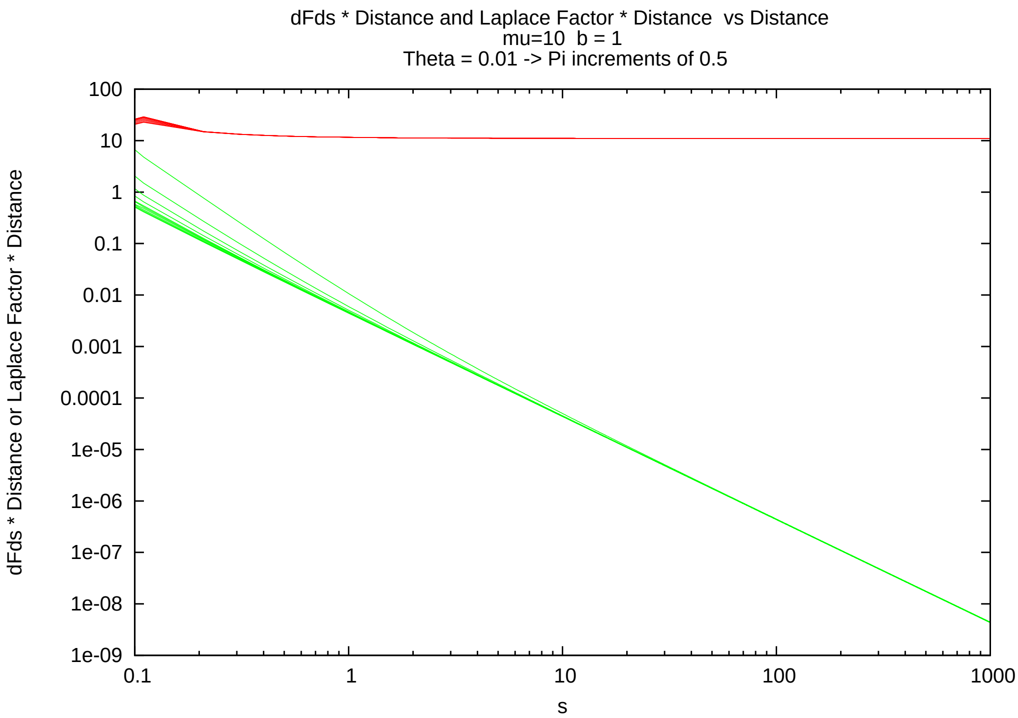

Figure 3: Quantities $R(s)s$ (upper curves) and $S(s)s$ (lower curves). Each is plotted for angles $\Theta = \{0.01, 0.51, 1.01, 1.51, 2.01, 2.51, 3.01\}$ radians.

where G_m is the approximation based on only the extremum path f_m. For relatively short times s, these additional paths should contribute relatively little, because a path with loops has more curvature and scattering attenuation than one that has no loops. But at long times the situation may be different. For the currently calculated $m = 0$ behavior, the factor Θ/s coming from f is small at long times. But for f_m, the factor at long times is $2\pi m/s$, and since m is summed over, there are values of m for which this is $O(1)$ or larger. It is entirely possible that, as the time grows, extremum paths with more and more loops become important as a mechanism for spreading energy throughout the volume. This could enhance the spatial spreading and bring $R(s)$ up in magnitude closer to $S(s)$, which would be more like the traditional diffusive regime. Further work addressing the role of looping extremum paths is needed.

At least two other approaches to approximating the FPI [7, 5] have produced diffusion in the traditional sense of (61). However, the approximation used by Perelman et al. [7] did not include an analysis of the extremum path around which to conduct the approximation, and so their approximation was effectively a small-angle approximation applied to a larger problem for which it may not be valid. Indeed, an extremely large numbers of small-angle scatterings is required to literally turn the light around and back to the illuminated boundary at large distances from the point source. In contrast, Miller [5] overlooks the phase function and directional effects altogether almost from the onset. Consequently, low-order Taylor expansions of the kernel naturally lead to spatially gaussian forms. It is notable that neither of these approaches yields the classic expression for the diffusivity constant [1, and references therein], $\mathcal{D} = ((1 - \langle\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}'\rangle)b + a)^{-1}$; rather it is taken, as needed, from the literature.

7. Conclusions

The Feynman Path Integral representation of the radiative transfer propagation kernel provides a framework for a systematic approximation process using standard approximation methods such as stationary phase and steepest descents. Applying these approximations produced an explicit expression for the spatial diffusion coefficient, and an asymptotic criterion for self-consistency of the approximation. The criterion is supported for small numbers of scattering lengths, but for large scattering lengths the situation is more complicated. The asymptotic criterion is "weakly" violated, i.e., it is not strongly supported, but also is not contradicted. One possibility is that further expansion of this approximation to higher orders may reveal better or additional asymptotic criteria.

The approximation process used here has a side effect of including some paths that should not be. By expanding the kernel in an expansion in small $|\mathbf{k}|$, the hard constraint in (8) is converted into a smoothed version of itself, allowing paths that are longer and shorter than s to be counted. This smearing effect on the kernel may be the source of the weak violation of the asymptotic criterion. A valuable future effort would focus on quantifying this error and finding a better approximation of the kernel that does not suffer from this smearing.

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A. The Feynman Path Integral Representation of the Kernel

A basic property of the Green's function is that it can be decomposed into the application of two Green's functions. Physically, this says that the propagation of light over an interval of time s can be decomposed into propagation over two successive time intervals s_0 and s_1, with $s = s_0 + s_1$. Mathematically this is the convolution of the Green's functions over the two time intervals:

$$G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') = \int d^3x_1\; d\Omega(\hat{\mathbf{n}}_1)\; G(s_1, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}_1, \hat{\mathbf{n}}_1)\; G(s - s_0, \mathbf{x}_1, \hat{\mathbf{n}}_1, \mathbf{x}', \hat{\mathbf{n}}'). \tag{72}$$

We can continue this decomposition and divide the total time interval into many subintervals. The full Green's function is a succession of convolutions of Green's functions for the N subintervals:

$$G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') = \int \prod_{i=1}^{N-1} d^3x_i\; d\Omega(\hat{\mathbf{n}}_i)\; \prod_{i=1}^{N} G(\Delta, \mathbf{x}_i, \hat{\mathbf{n}}_i, \mathbf{x}_{i-1}, \hat{\mathbf{n}}_{i-1}) \tag{73}$$

where Δ is the amount of time in each subinterval ($s = N\Delta$), $(\mathbf{x}_0, \hat{\mathbf{n}}_0) = (\mathbf{x}', \hat{\mathbf{n}}')$ and $(\mathbf{x}_N, \hat{\mathbf{n}}_N) = (\mathbf{x}, \hat{\mathbf{n}})$. This decomposition is exact for any N. No approximations have been applied at this stage.

To obtain the path integral, we will seek to subdivide the interval into infinitely many infinitismal subintervals, i.e., take the limit $N \to \infty$, $\Delta \to 0$ such that s remains fixed. In that limit, Δ is very small, so we can express the Green's function in each subinterval as a single scattering result:

$$\begin{aligned} G(\Delta, \mathbf{x}_i, \hat{\mathbf{n}}_i, \mathbf{x}_{i-1}, \hat{\mathbf{n}}_{i-1}) &= e^{-c\Delta} \delta\left(\mathbf{x}_i - \mathbf{x}_{i-1} - \hat{\mathbf{n}}_i \Delta\right) \\ &\times \left[\delta\left(\hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1}\right) + b\Delta P(\hat{\mathbf{n}}_i, \hat{\mathbf{n}}_{i-1})\right]. \end{aligned} \tag{74}$$

To be able to take the limit $\Delta \to 0$ with $N \to \infty$, we will need a way of representing the second factor, containing the angular dependence, in a different way. We can accomplish this using Fourier transforms. When the phase function $P(\hat{\mathbf{n}}, \hat{\mathbf{n}}')$ depends only on the inner product $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'$, then it can be expanded in Fourier modes as

$$P(\hat{\mathbf{n}}, \hat{\mathbf{n}}') = \int \frac{d^3p}{(2\pi)^3} \tilde{Z}(|\mathbf{p}|) \, \exp\left(i\mathbf{p} \cdot (\hat{\mathbf{n}} - \hat{\mathbf{n}}')\right). \tag{75}$$

This is not a true Fourier transform because the inverse transform does not exist. However, this pseudo-Fourier transform is all that is needed to proceed. Many phase functions in nature fulfill (75), including ocean water, tissues, atmosphere, and many types of clouds. For the situations that do not allow this specialization, Ref. [2] provides a generalization of this procedure.

In the regime $b\Delta \ll 1$, the angular factor becomes

$$\begin{aligned} \delta\left(\hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1}\right) + b\Delta P(\hat{\mathbf{n}}_i, \hat{\mathbf{n}}_{i-1}) &= \int \frac{d^3p}{(2\pi)^3} \, \exp\left(i\mathbf{p} \cdot (\hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1})\right) \\ &\times \exp\left(b\Delta \tilde{Z}(\mathbf{p})\right) + O\left(\Delta^2\right). \end{aligned} \tag{76}$$

This expression allows us to put together all of the subintervals and group the factors in a fairly compact way:

$$\begin{aligned} G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') &= e^{-cs} \int \prod_{i=1}^{N-1} d\Omega(\hat{\mathbf{n}}_i) \, \frac{d^3p_i}{(2\pi)^3} \, \exp\left(i \sum_{i=1}^{N} \mathbf{p}_i \cdot (\hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1})\right) \\ &\times \exp\left(\sum_{i=1}^{N} b\Delta \tilde{Z}(\mathbf{p}_i)\right) \\ &\times \int \prod_{i=1}^{N-1} d^3x_i \prod_{i=1}^{N} \delta\left(\mathbf{x}_i - \mathbf{x}_{i-1} - \hat{\mathbf{n}}_i \Delta\right). \end{aligned} \tag{77}$$

At this stage the spatial integrals can be evaluated exactly because they are a nested sequence of delta functions. They evaluate to

$$\int \prod_{i=1}^{N-1} d^3x_i \prod_{i=1}^{N} \delta\left(\mathbf{x}_i - \mathbf{x}_{i-1} - \hat{\mathbf{n}}_i \Delta\right) = \delta\left(\mathbf{x}_N - \mathbf{x}_0 - \sum_{i=1}^{N} \hat{\mathbf{n}}_i \, \Delta\right). \tag{78}$$

This sequence of substitutions and decompositions leaves us in good condition to take the continuum limit $\Delta \to 0$, $N \to \infty$. In this limit, the sums in the various terms become integrals. If we identify the continuous unit vector valued function $\hat{\beta}(s')$ with the direction vectors by $\hat{\beta}(s' = i\Delta) = \hat{\mathbf{n}}_i$, then the spatial delta function becomes in the continuum limit:

$$\delta\left(\mathbf{x}_N - \mathbf{x}_0 - \sum_{i=1}^{N} \hat{\mathbf{n}}_i \, \Delta\right) \to \delta\left(\mathbf{x} - \mathbf{x}' - \int_0^s \hat{\beta}(s') \, ds'\right). \tag{79}$$

Similarly, the factor containing the Fourier modes $\tilde{Z}$ of the phase function becomes an integral in the limit

$$\sum_{i=1}^{N} b\Delta \tilde{Z}(\mathbf{p}_i) \to \int_0^s b\tilde{Z}(\mathbf{p}(s')) ds'. \tag{80}$$

The last factor to convert to the continuum limit is the sum of the Fourier phase terms. By multiplying and dividing simultaneously by Δ, we can recognize that

$$\frac{\hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1}}{\Delta}$$

is a finite difference version of a derivative. So, in the continuum limit, it becomes $d\hat{\beta}(s')/ds'$ and the factor is

$$i\sum_{i=1}^{N-1} \mathbf{p}_i \cdot (\hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1}) = i\sum_{i=1}^{N-1} \mathbf{p}_i \cdot \frac{(\hat{\mathbf{n}}_i - \hat{\mathbf{n}}_{i-1})}{\Delta}\Delta \;\to\; i\int_0^s \mathbf{p}(s') \cdot \frac{d\hat{\beta}(s')}{ds'} ds'. \tag{81}$$

Assembling all of these results for limits, the final path integral expression for the Green's function is

$$\begin{aligned} G(s, \mathbf{x}, \hat{\mathbf{n}}, \mathbf{x}', \hat{\mathbf{n}}') \;&=\; e^{-cs}\int [d\hat{\beta}][d\mathbf{p}]\; \delta\left(\hat{\beta}(0) - \hat{\mathbf{n}}'\right)\; \delta\left(\hat{\beta}(s) - \hat{\mathbf{n}}\right) \\ &\times\; \delta\left(\mathbf{x} - \mathbf{x}' - \int_0^s \hat{\beta}(s')ds'\right) \\ &\times\; \exp\left(\int_0^s ds' \left(i\mathbf{p}(s') \cdot \frac{d\hat{\beta}(s')}{ds'} + b\tilde{Z}(\mathbf{p}(s'))\right)\right). \end{aligned} \tag{82}$$

We have also inserted delta function factors $\delta\left(\hat{\beta}(0) - \hat{\mathbf{n}}'\right)\; \delta\left(\hat{\beta}(s) - \hat{\mathbf{n}}\right)$ in order to explicitly enforce the "boundary conditions" imposed earlier that the direction of the path propagation has fixed values at the beginning and end of the path. Also, the integration measure $[d\hat{\beta}][d\mathbf{p}]$ is the "continuum limit" of the discrete multidimensional measure $\prod_i d\Omega(\hat{\beta}_i) d^3 p_i/(2\pi)^3$.

B. Derivation of the Approximate Feynman Integral Evaluation

The steps involved are an application of approximation methods that are used to derive the WKB approximation in quantum mechanics from the path integral representation of the propagator, and also applied in quantum field theory in the area on non-perturbative loop expansions. This procedure is also related to the methods for approximating ordinary integrals that are known as Laplace's Method, Stationary Phase, and Steepest Descents.

The goal is to complete an approximate evaluation of the path integral in (26). The steps are:

1. Find the function that extremizes the exponential in (26) such that the boundary conditions in the delta functions are satisfied. This function is the (28) with the arguments $f(s') = f_0(s') = \Theta s'/s$ and $g(s') = g_0(s') = 0$.
2. Express the integration function $\hat{\beta}(s')$ in terms of the extremizing function and a fluctuation around that solution. Hence we use $f(s') = f_0(s') + \delta f(s')$ and $g(s') = \delta g(s')$, where δf and δg are the fluctuations.
3. Taylor expand the exponential of (26) in the fluctuations, including all orders up to quadratic in δf and δg. The expansion of $A(\mathbf{p}_0)$ has no terms linear in the fluctuations, since we are expanding about the extremal function. It does have quadratic terms, and looks like

$$\begin{aligned} bN_p\int_0^s ds'\;(\,2\eta^2 R \;-\; 1/R\,) \;&=\; bsN_p\,(\,2\eta_0^2 R_0 \;-\; 1/R_0\,) \\ &+\; \frac{1}{2\mu bN_p}\int_0^s ds'\;\delta f(s')\left(-R_0\frac{d^2}{ds'^2}\right)\delta f(s') \\ &+\; \frac{1}{2\mu bN_p}\int_0^s ds'\;\delta g(s')\left(-R_0\frac{d^2}{ds'^2} \;-\; R_0\frac{\Theta^2}{2s^2}\right)\delta g(s') \end{aligned} \tag{83}$$

So the expansion of $A(\mathbf{p}_0)$ has only quadratic terms that are separated, i.e., there is no term proportional to $\delta f\;\delta g$. A similar expansion of the term

$$\int_0^s ds'\;\mathbf{k}\cdot\hat{\beta}(s')$$

produces

$$
\begin{aligned}
\int_0^s ds' \, \mathbf{k} \cdot \hat{\beta}(s') &= \int_0^s ds' \, \mathbf{k} \cdot \hat{\beta}_0(s') \\
&+ \int_0^s ds' \left(\mathbf{k} \cdot \left. \frac{d\hat{\beta}(s')}{df} \right|_0 \right) \delta f(s') \\
&+ \int_0^s ds' \left(\mathbf{k} \cdot \left. \frac{d\hat{\beta}(s')}{dg} \right|_0 \right) \delta g(s') \\
&+ \frac{1}{2} \int_0^s ds' \left(\mathbf{k} \cdot \left. \frac{d^2\hat{\beta}(s')}{df^2} \right|_0 \right) (\delta f(s'))^2 \\
&+ \frac{1}{2} \int_0^s ds' \left(\mathbf{k} \cdot \left. \frac{d^2\hat{\beta}(s')}{dg^2} \right|_0 \right) (\delta g(s'))^2 \\
&+ \int_0^s ds' \left(\mathbf{k} \cdot \left. \frac{d^2\hat{\beta}(s')}{dgdf} \right|_0 \right) \delta g(s') \, \delta f(s') \qquad (84)
\end{aligned}
$$

Evaluating each of the derivatives at the extremum,

$$\left. \frac{d\hat{\beta}(s')}{df} \right|_0 = -\hat{\mathbf{n}}' \sin(\Theta s'/s) + \hat{\mathbf{r}} \cos(\Theta s'/s) \qquad (85)$$

$$\left. \frac{d\hat{\beta}(s')}{dg} \right|_0 = \hat{\mathbf{t}} \qquad (86)$$

$$\left. \frac{d^2\hat{\beta}(s')}{df^2} \right|_0 = -\hat{\beta}_0(s') \qquad (87)$$

$$\left. \frac{d^2\hat{\beta}(s')}{dg^2} \right|_0 = -\hat{\beta}_0(s') \qquad (88)$$

$$\left. \frac{d^2\hat{\beta}(s')}{dgdf} \right|_0 = 0 \qquad (89)$$

So this term has a linear dependence on δf and δg, but it just so happens that the coefficient for the cross term $\delta f \; \delta g$ is zero. Assembling all of these into the Taylor expansion of the exponential in (26) leaves

$$
\begin{aligned}
A(\mathbf{p}_0) + i \int_0^s ds' \, \mathbf{k} \cdot \hat{\beta}(s') &\approx i \int_0^s ds' \left(\mathbf{k} \cdot \left. \frac{d\hat{\beta}(s')}{df} \right|_0 \right) \delta f(s') \\
&+ \frac{1}{2} \int_0^s ds' \, \delta f(s') \left(-\frac{R_0}{\mu b N_p} \frac{d^2}{ds'^2} - i\mathbf{k} \cdot \hat{\beta}_0(s') \right) \delta f(s') \\
&+ i \int_0^s ds' \left(\mathbf{k} \cdot \hat{\mathbf{t}} \right) \delta g(s') \\
&+ \frac{1}{2} \int_0^s ds' \, \delta g(s') \left(-\frac{R_0}{\mu b N_p} \frac{d^2}{ds'^2} - \frac{R_0}{\mu b N_p} \frac{\Theta^2}{2s^2} - i\mathbf{k} \cdot \hat{\beta}_0(s') \right) \delta g(s') \qquad (90)
\end{aligned}
$$

4. Evaluate the functional integrals over the fluctuation functions δf and δg. These functional integrals over δf and the one over δg can be carried out independently of each of other because the integrand separates into the product of two functionals, one depending only on δf and the other only on δg. Each one has the form

$$
\begin{aligned}
I(\mathbf{J}, M) &= \int [d\rho] \, \delta(\rho(0)) \, \delta(\rho(s)) \\
&\times \exp\left\{ -i \int_0^s ds' \, (\mathbf{k} \cdot \mathbf{J}(s')) \, \rho(s') \right\} \\
&\times \exp\left\{ -\frac{1}{2} \int_0^s ds' \, \rho(s') \, \Delta^{-1}_{M\mathbf{k}} \, \rho(s') \right\} \qquad (91)
\end{aligned}
$$

where, for the δf functional integral $M = 0$ and for the δg functional $M = \Theta/(s\sqrt{2})$. Equation (34) is the assemblage of all of these steps.

5. Since I is a gaussian functional integral, it can be evaluated exactly, and that evaluation leads to (35).

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